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Notes on shallow water extension of Miles Theory

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Abstract

We present results from a straightforward extension of Miles' theory of wind-wave generation towards shallow water. Both numerical results and an approximate analytical expression for the growthrate are obtained. The analytical expression forms the basis for a parametrization of wind wave growth in numerical wave prediction models.

The present wind input term in third generation models is based on the assumption that the wind-wave generation depends, regarding ocean wave properties, only on the phase speed of the waves. In this work it is shown that for the short waves ($kD > 1$, with k the wavenumber and D the local depth) this is indeed a good assumption, but for the long waves, with $kD << 1$, there will be an overestimate of wind input. Nevertheless, simulations with the energy balance equation for the duration limited case show that the present wind input term and the newly proposed windinput term give almost identical evolution of wave height and spectrum. This may be understood by noting that in equilibrium conditions $k_pD \simeq 1$ (with k_p the peak wavenumber), in other words in practical circumstances waves are short enough so that the difference between present and new wind input source function does not matter.

1 Introduction

In the mid 1980's Gerbrand Komen, Gao Quanduo and myself worked on the first version of the shallow water extension of the WAM model. One of the topics was, of course, to find an appropiate expression for the wind input source function. I extended Miles theory towards shallow water and we found that in a good approximation even in shallow water the wind input term may be parametrized in terms of the dimensionless phase speed c/u_* where c is the phase speed of the wave and u_* is the friction velocity in air. At least, the approximation was found to be adequate for fairly short waves with $kD > 1$. The reason that this approach works so well has to do with the fact that according to Miles theory the growthrate of the waves by wind is to a large extent determined by the curvature in the wind profile at the critical height z_c. Here, the critical height is determined by the condition that the wind speed $U_0(z)$ at z_c matches the phase speed of the surface gravity wave of interest: The condition $U_0(z_c) - c = 0$ expresses that there is a resonant interaction between the airflow around $z = z_c$ and the wave with phase speed c.

In 2012 I received a preliminary version of the paper of Montalvo *et al.* (2013) in which it was reported that there are considerable deviations from the WAM parametrization of the wind input source function. As I have lost all my notes on this subject it was decided to redo the original work in order to see whether the results from Montalvo *et al.* (2013) could be reconciled with the earlier results Gerbrand Komen and I found.

The programme of this note is therefore as follows. In §2 I extend Miles' theory towards shallow water waves and I discuss the differences with deep water theory. This is followed in §3 by a discussion of the numerical results and a derivation of an approximate solution, originally derived by Miles (1993) for deep water. However, Miles was fairly economical with the details of this derivation and therefore the full details are presented in the Appendix. The approximate expression for the wave growth is valid for small dimensionless critical height $\mu = k(z_c + z_0)$ only, where z_c is the critical height and z_0 the roughness length. As argued in §4, in deep water growth rates of the waves are underestimated for the long waves and therefore parameters in the approximate expression have been rescaled to match observed growth rates. This then gives the parametrization of wave growth that has been used for many years in the WAM model. This parametrization was designed in such a way that even in shallow water the growth rate only depends on the dimensionless phase speed c/u_*. However, in the present work the results of Montalvo *et al.* (2013) will be confirmed who showed that for really long waves with $kD << 1$ this approach is not adequate. Nevertheless, the approximate expression for the wave growth is still fine when the appropriate definition of the critical height in shallow water is used and when appropriate shallow water effects are taken into account. Unfortunately, in practice it turns out that simulations with the energy balance equation for the duration limited case show that the present wind input term and the newly proposed windinput term give almost identical evolution of wave height and spectrum. This is easily understood when it is realized that in equilibrium windsea conditions $k_pD \simeq 1$.

Nevertheless, in extreme circumstances there could be differences between the new and old approach, and since the new approach is physically more sound the resulting shallow water formulation was implemented in CY40R1 (November 2013).

2 Miles theory and shallow water effects.

The starting point is the treatment of Miles theory by Janssen (2004). Here, air and water are treated as one fluid. Growth rates of the waves by wind follow from the question under what conditions there is instability of the equilibrium

$$\mathbf{u}_0 = U_0(z)\mathbf{e}_x,\ \mathbf{g} = -g\mathbf{e}_z,$$
$$\rho_0 = \rho(z),\ p_0(z) = -g\int dz\,\rho_0(z), \tag{1}$$

where $\mathbf{e}_x$ and $\mathbf{e}_z$ are unit vectors in the x- and z-direction. Thus, we deal with a plane parallel flow whose speed U_0 and density ρ_0 only depends on height z.

In order to answer this question one uses the equations for an adiabatic fluid with an infinite sound speed and one linearizes these equations around the above equilibrium condition. For normal modes of the form $w = \hat{w}e^{i(kx-\omega t)}$, where in this case w is the vertical velocity, k is the wavenumber and ω is the unknown angular frequency, one finds the following Sturm-Liouville problem for the displacement of the streamlines $\psi = w/W$,

$$\frac{d}{dz}\left(\rho_0 W^2 \frac{d}{dz}\psi\right) - \left(k^2\rho_0 W^2 + g\rho_0'\right)\psi = 0. \tag{2}$$

Here, $W = U_0 - c$, with $c = \omega/k$ the phase speed of the waves and the prime denotes differentiation of an equilibrium quantity with respect to height z. The prominent role of the critical layer is evident through the Doppler-shifted velocity W. The equation for the perturbed streamlines is subject to the boundary condition of vanishing displacement at infinite height and at finite depth D,

$$\psi \to 0,\ z \to \infty$$
$$\psi(z = -D) = 0. \tag{3}$$

The boundary value problem (2)-(3) determines, in principle, the real and imaginary part of the complex phase speed $c = \omega/k$, giving the growth rate $\gamma = \Im(\omega)$ of the waves.

Whether there is instability or not can only be decided by solving the boundary value problem. This will be done for the special case of surface gravity waves. This case follows by choosing an appropriate density profile, i.e. the density profile is chosen constant in air and water with a jump at the interface at $z = 0$. As the air density ρ_a is much smaller than the water density ρ_w, there is a small parameter $s = \rho_a/\rho_w \approx 10^{-3}$ in the problem allowing to construct an approximate solution of the eigenvalue problem (2)-(3).

In water we assume that effects of the current may be neglected and that the water density is a constant, hence (2) becomes for $z < 0$

$$\frac{d^2}{dz^2}\psi_w = k^2\psi_w, \tag{4}$$

and with the boundary condition $\psi(z = -D) = 0$ the solution becomes

$$\psi_w = \psi_0 \frac{\sinh k(z+D)}{\sinh kD}, \tag{5}$$

where ψ_0 is arbitrary. At the air-sea interface the boundary condition is now that ψ be continuous. The second condition follows from an integration of (2) from just below the surface, $z = -0$, to just above the surface, $z = +0$. Note that at $z = 0$ the density profile shows a jump so that $\rho_0' = (\rho_a - \rho_w)\delta(z)$ where $\delta(z)$ is the Dirac delta function. The result is

$$\rho_0 W^2 \frac{d}{dz}\psi\Big|_{-0}^{+0} = \int_{-0}^{+0} dz\ \left[\rho_0 k^2 W^2 + g\rho_0'\right]\psi. \tag{6}$$

Since in the limit only the integral involving ρ_0' gives a contribution, we obtain, using (5), the following dispersion relation for the phase speed of the waves

$$\frac{c^2}{c_0^2} = \frac{(1-s)}{1 - sZ\psi_a'(0)/k},\ s = \frac{\rho_a}{\rho_w},\ Z = \frac{\omega_0^2}{gk}, \tag{7}$$

with $\omega_0^2 = gk\tanh kD$ the dispersion relation for free gravity waves in shallow water while c_0 is the corresponding phase speed of the waves. Furthermore, without loss of generality, we have taken the amplitude $\psi_0 = 1$ as we are dealing with a linear problem. Once the vertical gradient of the displacement of the streamlines in the air (i.e. $\psi_a'(0)$) is known we can establish the growth rate of the waves. This follows in principle from the application of (2)-(3) to the air. Taking a constant air density one finds for ψ_a the simplified problem

$$\begin{aligned} \frac{d}{dz}\left(W^2\frac{d}{dz}\psi_a\right) &= k^2W^2\psi_a, \\ \psi_a(0) &= 1; \\ \psi_a \to 0,\ z &\to \infty. \end{aligned} \tag{8}$$

Here, $W = U_0(z) - c$ is still unknown as c follows from Eq. (7), which shows that the phase speed c depends on the solution ψ_a but this dependence is weak as the density ratio s is small. In other words, in the absence of air the usual dispersion relation for free gravity waves, i.e. $c^2 = c_0^2$, follows. The effect of air on the surface gravity waves is small since $s \ll 1$ and therefore the dispersion relation can be solved in an approximate manner with the result

$$c = c_0 + sc_1 + ..., \tag{9}$$

with $c_0^2 = g\tanh kD/k$ and $c_1 = \frac{1}{2}c_0(Z\psi_a'/k - 1)$. As a result, the problem (8) now reduces to

$$\begin{aligned} \frac{d}{dz}\left(W_0^2\frac{d}{dz}\psi_a\right) &= k^2W_0^2\psi_a, \\ \psi_a(0) &= 1, \\ \psi_a \to 0,\ z &\to \infty. \end{aligned} \tag{10}$$

where $W_0 = U_0 - c_0$ is now known, and as a consequence the solution to the differential equation is simplified considerably. However, the limit of vanishing c_1 in the expression for the Doppler-shifted velocity has to be taking with care, in particular when integrating expressions involving the inverse of W_0. This is discussed in more detail in Janssen (2004), and as a rule, for growing solutions, the integration contour needs to be indented below the singularity. In addition, we now have an explicit expression for the growth rate γ_a of the amplitude of the waves. With $\Im$ the imaginary part we find

$$\frac{\gamma_a}{\omega_0} = s\Im\left(\frac{c_1}{c_0}\right) = \frac{sZ(kD)}{2k}\Im(\psi_a') = \frac{sZ(kD)}{4k}\mathscr{W}(\psi_a, \psi_a^*)\Big|_{z=0} \tag{11}$$

where the Wronskian $\mathscr{W}$ is given by

$$\mathscr{W}(\psi_a, \psi_a^*) = -i\left(\psi_a'\psi_a^* - \psi_a\psi_a'^*\right). \tag{12}$$

As discussed in Janssen (2004) the Wronskian has a physical interpretation as it is directly connected to the wave-induced stress τ_w,

$$\tau_w = -\langle\delta u \delta w\rangle = -\frac{i}{k}\left(w^* w' - w w'^*\right) = \frac{W^2}{k}\mathscr{W}(\psi_a, \psi_a^*). \tag{13}$$

In summary, in order to obtain the growthrate of waves by wind one needs to solve the boundary value problem (10) and then ψ_a' is used in (11) which then immediately gives the growthrate γ_a. Before in the next Section two possible methods of solving the differential equation (10) are discussed, we mention here that shallow water effects enter the problem in two ways. The first one is through the factor $Z = \omega_0^2/gk$ in the expression for the growth rate. Clearly, when Z is smaller than 1 there are deviations from the deep-water dispersion relation. In fact, in the shallow water limit $kD \to 0$ one finds that $Z \to kD$. Hence, for shallow water waves the growthrate of the waves by wind is expected to become small, provided the displacement of the streamlines in air remains finite in the shallow water limit.

Shallow water effects also occur through the Doppler-shifted velocity $W_0 = U_0 - c_0$. Whilst in deep water the phase speed is ever increasing with decreasing wavenumber, in sharp contrast to this, in shallow water the phase speed c_0 reaches a maximum given by $c_0^{max} = \sqrt{gD}$ for very long waves. This will have a pronounced impact on the magnitude of the critical height and on the growth rate, which will increase as explained in the next Section. However, combining now the two effects it turns out that the depth dependence of Z is overwhelming with the result that for the same wavenumber the growthrate of shallow water waves is smaller than the corresponding growth rate for deep water, while in the limit of shallow water waves the growth rate vanishes. This will be discussed in more detail in the next section.

3 Numerical and analytical solution.

Before we present numerical and analytical results, it is remarked that it is rather common to use the vertical component of the wave-induced velocity instead of the displacement of the streamlines $\psi \sim w/W$. In terms of the normalised vertical velocity $\chi = w/w(0)$, the eigenvalue problem (10) becomes

$$\begin{aligned} W_0\left(\frac{d^2}{dz^2} - k^2\right)\chi &= W_0''\chi, \\ \chi(0) &= 1, \\ \chi \to 0,\ z &\to \infty. \end{aligned} \tag{14}$$

and the growth rate of the wave is given by

$$\frac{\gamma_a}{s\omega_0} = \frac{Z(kD)}{4k} \mathscr{W}(\chi,\chi^*)\Big|_{z=0}, \tag{15}$$

where the Wronskian is now given by $\mathscr{W} = -i(\chi'\chi^* - \chi\chi'^*)$.

Regarding (14) we remark that the differential equation, known as the Rayleigh equation, has a singularity at $W_0 = U_0 - c_0 = 0$. Since $W_0 = 0$ defines the critical height z_c (i.e. the height where the phase speed of the wave matches the wind speed) it is now clear that the resonance at the critical height plays a special role in the problem of wind-wave generation. The role of the critical height becomes even clearer when the Wronskian $\mathscr{W}$ is evaluated. By means of the Rayleigh equation it may be shown that the Wronskian is independent of height except at the critical height where it may show a jump. Explicitely one finds that for $z > z_c$ the Wronskian vanishes while for $z < z_c$ one finds

$$\mathscr{W} = -2\pi \frac{W_{0c}''}{\mid W_{0c}' \mid} \mid \chi_c \mid^2,$$

and therefore the growth rate of the waves becomes

$$\frac{\gamma_a}{s\omega_0} = -\frac{\pi Z(kD)}{2k} \frac{W_{0c}''}{\mid W_{0c}' \mid} \mid \chi_c \mid^2 . \tag{16}$$

This is Miles' classical result for the growth of surface gravity waves due to shear flow. From (16) we obtain the well-known result that only those waves are unstable for which the curvature $W_0'' = U_0''$ of the wind profile at the critical height is negative. This is the case, for example, for a logarithmic profile. Therefore, the critical height z_c plays a key role in the problem of the generation of ocean waves by wind as the air at z_c height enjoys a resonant interaction with the gravity wave with phase speed c.

In order to solve the boundary value problem (14), we finally have to specify the shape of the wind profile. In case of neutrally stable conditions (no density stratification by heat and moisture) the wind profile has a logarithmic height dependence. Hence,

$$U_0(z) = \frac{u_*}{\kappa} \log(1 + z/z_0), \tag{17}$$

which follows from the condition that the momentum flux in the surface layer is a constant for steady conditions. Recall that $\kappa = 0.40$ is the von Kármán constant which is supposed to be a universal constant, the friction velocity u_* is a measure for the momentum flux in the surface layer and the roughness length z_0 is a parameter which reflects the momentum loss at the sea surface. It is given by the Charnock relation

$$z_0 = \alpha_{\mathrm{CH}} u_*^2 / g, \tag{18}$$

with α_{CH} the Charnock parameter. Although in the present linear treatment it is assumed that the Charnock parameter α_{CH} is a constant [1], there are arguments why α_{CH} is not a constant but depends on the sea state.

For given wind profile (17) the boundary value problem (14) may now be solved. Originally, Miles (1957) applied a variational approach to obtain an approximate solution. However, when compared to the numerical results of Conte and Miles (1959) this approximation gave growth rates which were too large by a factor of three. Prompted by asymptotic matching results of van Duin and Janssen (1992), Miles (1993) revisited this problem and realized that the overestimation was caused by the neglect of

[1] we take $\alpha_{\mathrm{CH}} = 0.0144$, because this was the typical value in the bight of Abaco experiment of Snyder *et al.* (1981).

some higher order terms. The improved approximation gave good agreement with Conte and Miles (1959). However, the approximation of Miles (1993) is formally only valid for 'slow' waves which have a small critical height. This will be illustrated by comparing the results of the Miles approximation with the numerical results. Furthermore, we study shallow water effects on the growth rate of the waves by wind and we extend the range of validity of the Miles' approximation towards shallow water.

3.1 numerical results.

Before we discuss the numerical results of the growth rate, dimensionless quantities are introduced in order to see which parameters determine the problem. This scaling behaviour can be achieved in several manners and we discuss two of them, one in the context of the numerical solution and one in the context of the approximate solution.

Since we are dealing with gravity waves it is natural to use the acceleration of gravity g as a basic scaling quantity. Judging from the equilibrium form of the wind profile (17) the other basic scaling quantity is the friction velocity u_*. Thus, we scale velocities with u_* and in agreement with the Charnock relation (18) lengths scale with u_*^2/g, hence

$$z_* = gz/u_*^2, z_{0*} = gz_0/u_*^2, D_* = gD/u_*^2$$
$$c_* = c/u_*, U_{0*} = U_0/u_*, \tag{19a}$$

while the dimensionless wavenumber becomes

$$k_* = ku_*^2/g. \tag{19b}$$

As a consequence, the boundary value problem (14) has in terms of these dimensionless quantities the same form, while from (17)- (18) the dimensionless wind profile becomes

$$U_{0*} = \frac{1}{\kappa}\log(1+\frac{z_*}{z_{0*}}). \tag{20}$$

For a given wave characterized by its dimensionless phase speed c_* we can solve for the dimensionless growth rate γ/ω_0 of the energy of the waves, which is twice the growth rate of the amplitude of the waves,

$$\gamma/\omega_0 = 2\gamma_a/\omega_0. \tag{21}$$

The dimensionless growth rate depends in general on the dimensionless phase speed c_* and on two dimensionless parameters, namely the dimensionless roughness length z_{0*} and the dimensionless depth parameter D_*. However, remarkably, with Charnock's relation we have $z_{0*} = \alpha_{CH}$ which for the moment is regarded as a constant, independent of u_* and of sea state. Therefore, for a neutrally stable airflow with a logarithmic wind profile, the growth rate γ/ω_0 only depends on c_* and on the parameter D_*.

Following scaling arguments of Miles (1957) the growth rate of the waves can be written in the general form

$$\gamma/\omega_0 = s\beta\frac{u_*^2}{c^2}, \tag{22}$$

where β is the so-called Miles parameter. In Fig. 1 a plot is shown of the Miles parameter as function of the dimensionless phase speed c/u_* for the deep water case ($D_* = 30,000$) and a shallow water case

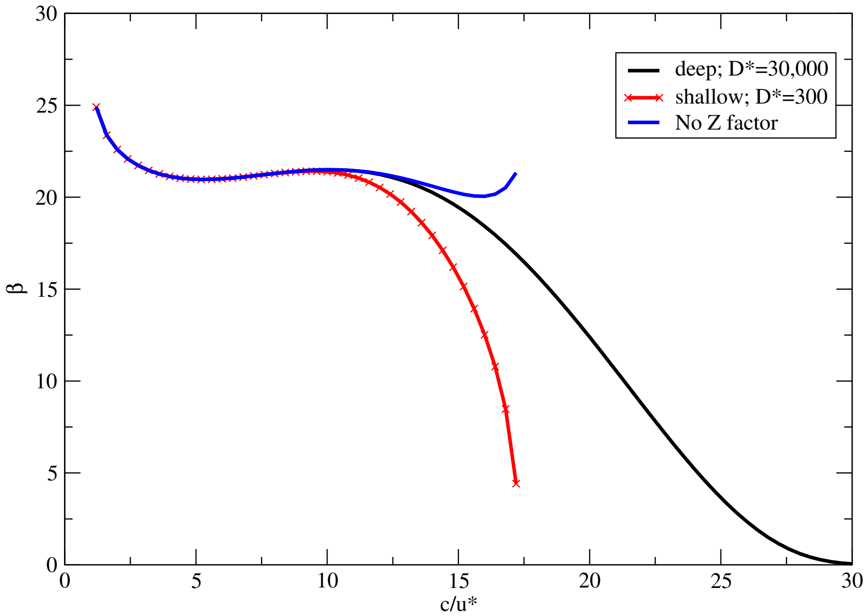

Figure 1: The Miles' parameter β as function of the dimensionless phase speed c/u_* for a Charnock parameter of $\alpha_{\mathrm{CH}} = 0.0144$. Shown are the deep water case and a shallow water case with $gD/u_*^2 = 300$. Finally, also the result of a calculation is shown where the factor $Z = \omega^2/(gk)$ is set to one.

with $D_* = 300$. The growth rate γ is obtained using Eq. (16) and (21), where the vertical velocity at the critical height is obtained from the Rayleigh equation (14) using the method of Conte and Miles (1957). For further comments see Janssen (2004). From the comparison between the deep and shallow water case it is evident that shallow water waves have a reduced energy input by wind. It should also be clear that in shallow water the dimensionless phase speed is limited by depth. In fact, $c_*^{max} = \sqrt{D_*} \approx 17.32$ and for this reason the shallow water curve stops at that dimensionless phase speed.

It is of interest to try to understand why the growth rate of the waves reduces so dramatically near the limiting phase speed. For this reason, we repeated the calculation for the growth rate where we set $Z = 1$ in the expression for γ and we only retained shallow water effects in the Rayleigh equation. In that case for shallow water waves the growth rate increases somewhat compared to the deep water case. Therefore, it is concluded that the main reason for the dramatic reduction of the growth rate of waves by wind is the introduction of the Z-factor in Eq. (16). A deeper explanation of all this will be given in the next subsection after the analytical approach has been discussed.

3.2 Analytical approach.

In order to introduce a method to obtain an approximate solution of the Rayleigh equation we perform here a slightly different scaling transformation which will give a considerable simplification in particular when one wishes to understand the results. To that end we scale the height variable with the wave number k and we shift the origin of the vertical coordinate by an amount of the dimensionless roughness, $y_0 = kz_0$,

$$y = k(z + z_0)$$

and the boundary value problem (14) becomes

$$W\left(\frac{d^2}{dy^2}-1\right)\chi = W''\chi,$$
$$\chi(y_0)=1, \tag{23}$$
$$\chi \to 0,\ y\to\infty.$$

where the prime now denotes differentiation of an equilibrium quantity with respect to y. Here, $W = U_0(y)-c_0$ with wind profile

$$U_0 = \frac{u_*}{\kappa}\log\left(\frac{y}{y_0}\right), \tag{24}$$

and, as can be seen the choice of vertical coordinate y simplifies the wind profile. The growth rate given in Eq. (16) then becomes

$$\frac{\gamma}{s\omega_0} = -\pi Z(kD)\frac{W_c''}{\mid W_c'\mid}\mid\chi_c\mid^2.$$

Finally, for a logarithmic wind profile the dimensionless critical height $\mu = y_c$ follows from the condition $U_0(\mu)=c_0$ and is given by the expression

$$\mu = y_c = y_0 e^{\kappa c/u_*}. \tag{25}$$

We need to study the properties of the above expression for μ in order to understand the differences between deep and shallow water waves.

An alternative expression for the growthrate follows from Eq. (15). In terms of the present vertical coordinate y one finds

$$\frac{\gamma}{s\omega_0} = \frac{Z(kD)}{2}\mathscr{W}(\chi,\chi^*)\Big|_{y=y_0}, \tag{26}$$

with

$$\mathscr{W}(\chi,\chi^*) = -i\left(\chi^*\frac{d\chi}{dy}-\chi\frac{d\chi^*}{dy}\right)$$

The last form is more convenient for the present analysis.

In order to obtain the growth rate γ we have to solve for the Rayleigh equation, but with a logarithmic wind profile this cannot be done exactly. Nevertheless, approximate solutions may be found. For example, for large y one may ignore the curvature term so that to a good approximation it is found that

$$\left(\frac{d^2}{dy^2}-1\right)\chi \approx 0 \tag{27}$$

and for the boundary condition of vanishing χ at infinity the so-called outer solution becomes

$$\chi \approx Ae^{-y},\ y\to\infty, \tag{28}$$

where A is an unknown constant. On the other hand, for small y the second term in the Rayleigh equation may be ignored hence

$$W\frac{d^2}{dy^2}\chi - \chi\frac{d^2}{dy^2}W = 0, \tag{29}$$

which has the general solution

$$\chi \approx aW\left(1+b\int_{y_0}^{y}\frac{\mathrm{d}\eta}{W^2(\eta)}\right), \tag{30}$$

where a and b are obtained from the boundary conditions. The above solution which holds for small y is called the inner solution. Then, the constant a immediately follows from the boundary condition $\chi(y_0)=1$, hence $a=1/W(y_0)$. However, it is not straightforward to determine the constant b from the boundary condition at infinity. One reason is that the solution (30) is only valid for small y, while the solution (28) is valid for large y. Another reason is that formally the second independent solution of (30) has a singularity and a branch cut needs to be introduced in order to make it single-valued.

In the Appendix it is shown how by means of the method of asymptotic matching the parameter b may be obtained. Basically, the small y limit of the outer solution (28) is matched to the large y limit of the inner solution (30) resulting in algebraic equations for A, a and b. As a result one finds

$$b=-\frac{u_*^2}{\kappa^2}\log^2\left(\frac{y_c}{\lambda}\right)\frac{1}{1+i\Gamma}, \tag{31}$$

where

$$\Gamma=\pi y_c\log^2\left(\frac{y_c}{\lambda}\right),\ \text{and}\ \lambda=\frac{1}{2}e^{-\gamma_E}\approx 0.281, \tag{32}$$

with $\gamma_E=0.5771$ Euler's constant.

It is emphasized that this solution has a restricted validity because it is assumed that the inner region, with $y=\mathscr{O}(y_c)$, and outer region, with $y=\mathscr{O}(1)$, are distinct. This can only be achieved if $y_c\ll 1$. In deep water, therefore, where y_c increases for increasing phase speed the above approximation is expected to fail. In shallow water where there is a limiting phase speed this needs not to be the case and the above solution with b given by Eq. (31) works remarkably well as will be seen in a moment.

Now, in order to evaluate the growth rate the Wronskian $\mathscr{W}$ is required. Using Eq. (30) one finds

$$\mathscr{W}=\frac{2\pi}{\kappa^2}y_c\log^4\left(\frac{y_c}{\lambda}\right)\frac{u_*^2}{c^2} \tag{33}$$

and the analytical expression for the growth rate follows at once by substituting (33) into (26),

$$\gamma/\omega_0=s\beta\frac{u_*^2}{c^2},\ \beta=\frac{\pi}{\kappa^2}Z(kD)y_c\log^4\left(\frac{y_c}{\lambda}\right),y_c\leq\lambda. \tag{34}$$

In deep water this result agrees with Miles (1993).

Let us compare the analytical approximation with the numerical results for deep and shallow water. In Fig. 2 the Miles' parameter β is plotted as function of the dimensionless phase speed c/u_* for a Charnock parameter of $\alpha_{\mathrm{CH}}=0.0144$. Shown are numerical results for the deep water case and a shallow water case with $gD/u_*^2=300$. For comparison corresponding results using the analytical approach, Eq. (34) are show as well. Considering the deep water case first, it is clear that the analytical result (34) is a fair approximation to the numerical results [2] for the slow waves, but for the fast, long waves, which have a dimensionless critical height of the order 1, the analytical result clearly fails. On the other hand, in shallow water, the agreement between numerical results and (34) is much more impressive.

[2] in particular realizing that the original approximation by Miles (1957) was off by a factor of 3. The present improvement is entirely caused by the factor λ in the log term of (34). The λ factor follows from higher order matching as explained in detail in the Appendix.

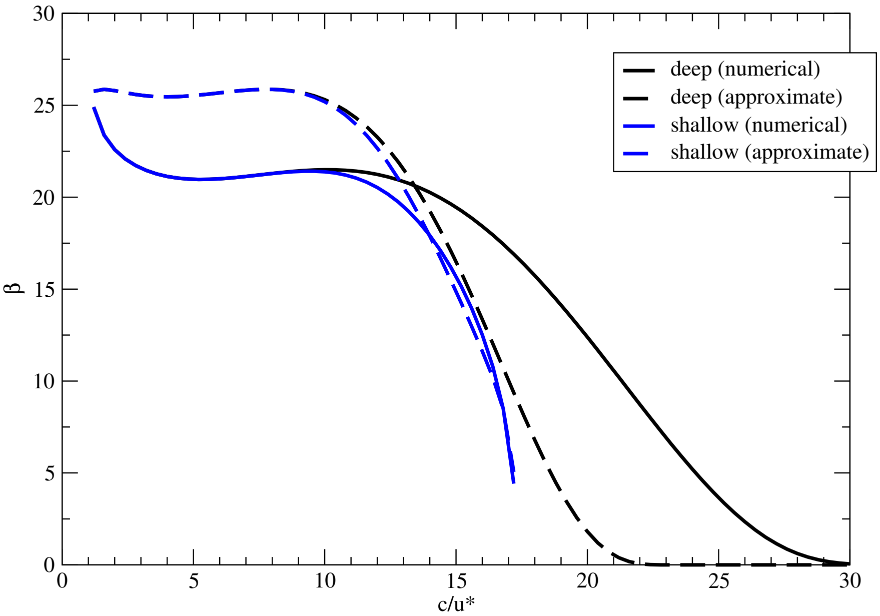

Figure 2: The Miles' parameter β as function of the dimensionless phase speed c/u_* for a Charnock parameter of $\alpha_{\rm CH} = 0.0144$. Shown are numerical results for the deep water case and a shallow water case with $gD/u_*^2 = 300$. For comparison corresponding results using the analytical approach, Eq. (34) are show as well.

In order to appreciate this point somewhat better it is important to realize that according to (34) the Miles parameter only depends on two parameters, namely dimensionless depth and the dimensionless critical height y_c. Disregarding the Z-factor for the moment it is of interest therefore to study the different behaviours of the critical height for deep and shallow water waves. The critical height follows from Eq. (25) and is given by the expression

$$y_c = kz_0 e^{\kappa c/u_*}.$$

While for deep water waves the critical height increases almost exponentially when the waves get longer, in sharp contrast, for shallow water waves the critical height remains finite in the long wave limit and even vanishes as the phase speed approaches c_*^{max}. As the analytical approximation is valid for small dimensionless critical height y_c it should be clear now that in shallow water (34) is a fair approximation to the numerical results even for the long waves. Now including the Z-factor and realizing that in practice the dependence of the Miles'parameter on the critical height is fairly weak it is now evident that the Z-factor controls the behaviour of the Miles parameter for really shallow water waves.

4 Parametrizing the wind input term including shallow water effects.

Here, I describe my efforts to obtain a simple parametrization of the wind input source function using the analytical approximation of Miles critical layer theory (see Eq. (34)). This parametrization should incorporate the dependence of the growth rate on the dimensionless phase speed, the dimensionless roughness length and the dimensionless depth. There are however a few problems with the analytical approach. Comparing numerical results with (34) we have seen that for the long waves the asymptotic

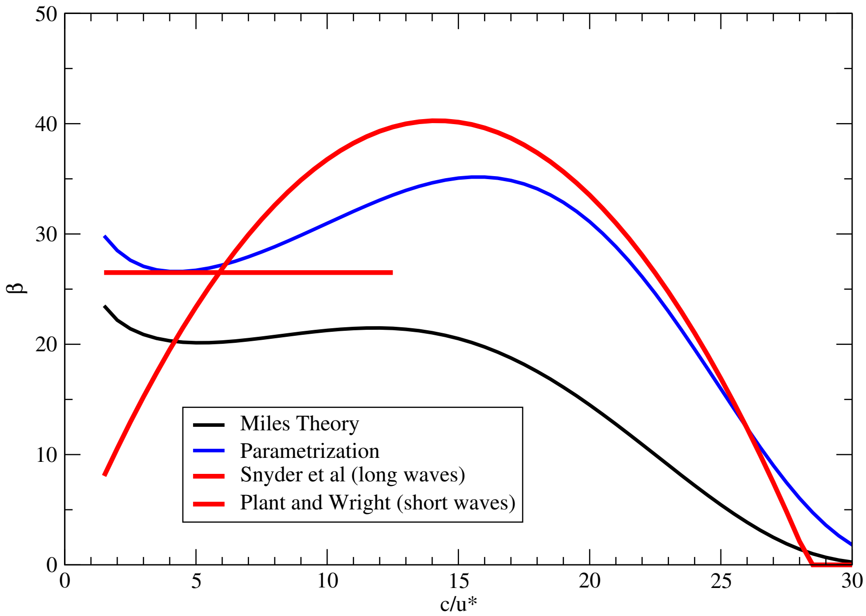

Figure 3: The Miles' parameter β as function of the dimensionless phase speed c/u_* for a Charnock parameter of $\alpha_{\mathrm{CH}} = 0.0144$. Shown are numerical results using Miles theory, from analytical approach with adjusted coefficients, and two empirical fits, one based on Snyder et al. (1981) valid for fast waves, and one from Plant (1982), which is a simple constant $\beta = 26$, valid for slow waves $c/u_* < 12$.

result underestimates wave growth. Of course, it is even more important to compare with fits of the growthrate based on observations of wave growth by wind. Probably the most reliable empirical fit is from Snyder *et al.* (1981). This fit is a good approximation to the growthrate of the long waves. It reads

$$\frac{\gamma}{\omega} = 0.25s\left(\frac{U_5}{c} - 1\right) \tag{35}$$

where U_5 is the wind speed at 5 metre height. As in this note the friction velocity is the quantity which is assumed to be known, the wind speed U_5 is obtained using the logarithmic profile (17), where the Charnock parameter $\alpha_{\mathrm{CH}} = 0.0144$ so that for a friction velocity u_* of 0.2 m/s the drag coefficient is about 1.2410^{-3}. This is in reasonable agreement with observed drag from the bight of Abaco experiment (Snyder *et al.*, 1981). The corresponding Miles parameter is plotted in Fig. 3 and is a valid approximation to the observations for $c/u_* > 10$. For the short waves with $c/u_* < 10$ I will use the empirical fit of Plant (1982), which is basically Eq. (34) with $\beta \approx 26$. Now comparing the numerical results shown in Fig.3 with the emprical fits to the observations, it is clear that for the short waves Miles theory gives a good agreement with observation, but that in particular in the phase speed range $12 < c/u_* < 30$ Miles critical layer theory underestimates observed growth somewhat.

For this reason we have rescaled parameters in the analytical formula by replacing $\lambda = 0.281$ by $\lambda = 1.$, and by replacing π by the factor 1.2. In addition, in the formula for the critical height, we have shifted u_*/c by a factor $z_\alpha = 0.006$. As a result the following parametrization of the growth rate in the wind input source function is suggested:

$$\gamma/\omega_0 = s\beta\frac{u_*^2}{c^2},\ \beta = \beta_{max}\kappa^2 Z(kD) y_c \log^4(y_c), y_c \leq 1, \tag{36}$$

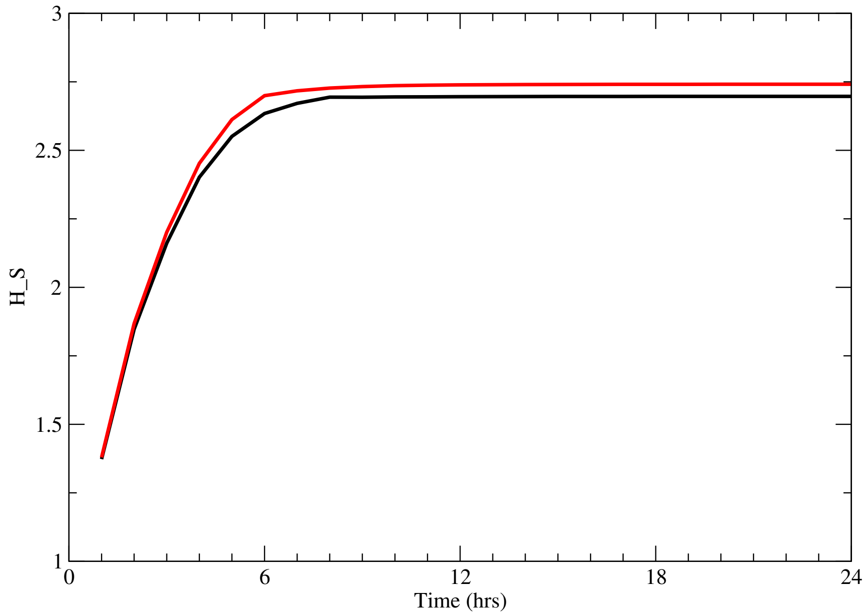

Figure 4: Significant wave height H_S as function of duration for a constant wind of 18.45 m/s and a depth of 10 m. The black curve is for the 'new' wind input source function and the red curve is for the 'old wind input term.

where

$$\beta_{max} = 1.2,\ y_c = kz_0 e^{\kappa/x},\ x = u_*/c + z_\alpha,\ z_\alpha = 0.006.$$

The parametrization is shown in Fig. 3 as well and a good agreement with the empirical fits is to be noted.

The above parametrization has been used since WAM cy4. However, in the first implementation a much higher value for z_α was chosen, namely $z_\alpha = 0.011$. This value was chosen in order to have an amount of dissipation which was similar to the WAM cy3 model. But this choice gives, in comparison with observed wave growth, rise to too much low-frequency energy. This was also found in extensive verification efforts done by Jean Bidlot. It was therefore decided to reduce the wind input to the long waves by reducing z_α from 0.011 to 0.008. As a consequence wave dissipation was reduced as well. This, as will be reported elsewhere, has given rise to considerable improvements in our ability to forecast signficant wave height and wave periods. Nevertheless, the parameter z_α is still slightly too high, so more work in this direction is required.

5 Energy balance equation in shallow water.

As a final check I have run with my private version of the ECWAM model a shallow water case with the old and the new parametrization of the wind input source function. The run simulates duration-limited growth of a wind sea with a constant wind speed of 18.45 m/s in water of 10 m depth.

The 'old' parametrization of the wind source function is very similar to Eq. (36) except the Z-factor is set to 1 while the critical height is regarded as a function of phase speed only, i.e. wave number is replaced

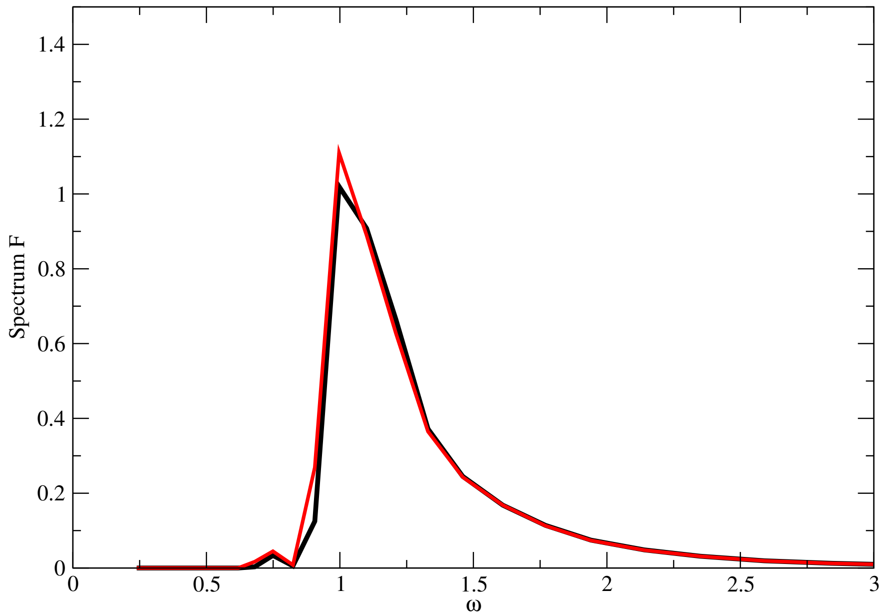

Figure 5: Angular frequency spectrum after 24 hrs for 'new' (black curve) and 'old' (red curve) wind input source function.

everywhere using the deep-water dispersion relation,

$$y_c^{old} = \frac{gz_0}{c^2} e^{\kappa/x}, x = u_*/c + z_\alpha, z_\alpha = 0.008.$$

In this fashion, plotted as function of the dimensionless phase speed c/u_*, the relative growth rates in deep and shallow water are identical. On the other hand, with the revision given in (36), which agrees with what has been found in Figs. 1 and 2, growth rates are considerably reduced for the long waves. It is therefore of interest to see how much the change in wind input source function affects for shallow water the growth of wind sea by a constant wind.

Results for wave height and one-dimensional angular frequency spectrum are shown in the Figs. 4 and 5. These results do suggest that there is only little impact to be expected of this change in wind input term. The reason is that for equilibrium wind waves the characteristic dimensionless depth parameter k_pD is about 1, therefore the reduction of the wind input is relatively small. Nevertheless, in more complicated circumstances, such as a sloping bottom and rapidly varying winds there might be bigger effects, but this still needs to be tested.

6 Conclusions.

Prompted by the work of Montalvo *et al.* (2013) I have revisited Miles theory of the growth of wind-waves in shallow water. In agreement with their work I find that for very shallow water, i.e. $k_pD < 1$, there is indeed a considerable impact of finite depth on the growth rate of the waves. It is straightforward to extend the standard wind input formulation of WAM cy4 towards shallow water cases. However, relative little impact is found on the evolution of the sea state in shallow water. Presumably, the reason for this is that in equilibrium the sea state in shallow water has a peak wavenumber k_p that approximately satisfies the condition $k_pD \approx 1$, in other words in practice shallow water effects on wind-wave growth are relatively small.

Acknowledgement The author acknowledges useful and stimulating discussions with Miguel Onorato and Jean Bidlot. He thanks Roger Grimshaw for confirming the mathematical results obtained in the Appendix.

Appendix.

A Approximate solution of the Rayleigh Equation.

The following results have been obtained after a long-standing collaboration I had with Cees van Duijn in the 1990's. We usually worked in terms of the stream function Ψ defined in such a way that $u = -\partial\Psi/\partial z$ and $w = \partial\Psi/\partial x$. Therefore, the transition from the Fourier transform of the vertical velocity to the Fourier transform of the stream function is straightforward since $\hat{w} = ik\hat{\Psi}$. As in the main text the hats will be dropped.

A.1 Statement of the problem.

Our starting point is the boundary value problem for the normalized vertical velocity χ given in Eq. (23) with wind profile (24). The stream function Ψ is introduced through $w = ik\Psi$, and the normalization by $w(0)$ is ignored. Next, we shift the origin of the vertical coordinate y by an amount y_0 which results in a simpler wind profile and the boundary condition at $y = 0$ is now applied at height y_0. Thus, we search for an approximate solution of the following boundary value problem:

$$\begin{aligned} W\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - 1\right)\Psi &= W''\Psi, \\ \Psi(y_0) &= -c, \\ \Psi \to 0,\ y &\to \infty. \end{aligned} \tag{A1}$$

where the prime now denotes differentiation of an equilibrium quantity with respect to $y = k(z + z_0)$, with z is the distance from the water surface. Here, $W = U_0(y) - c$ with wind profile

$$U_0 = \frac{u_*}{\kappa}\log\left(\frac{y}{y_0}\right), \tag{A2}$$

and the dimensionless roughness is given by $y_0 = kz_0$.

In order to make progress the following important relations are introduced. Define the scale velocity

$$V = U_0(\alpha), \tag{A3}$$

where α will be determined in such a way that the convergence of the perturbation expansion is optimal. Velocities will be scaled with V and we regard

$$\varepsilon = \frac{u_*}{V} = \frac{\kappa}{\log(\alpha/y_0)} \tag{A4}$$

as a small parameter. For a dimensionless phase speed $c_* = c/u_* = 5$ and a Charnock parameter $\alpha_{CH} = 0.0144$ the small parameter $\varepsilon \approx 0.05$. Another small parameter in the problem is y_0. This parameter is compared to ε exponentially small (for the example at hand one has $y_0 \approx 0.0006$), and therefore effects of the order y_0 will be ignored in the treatment that follows.

Scaled with V the wind profile becomes

$$U_0 = 1 + \frac{\varepsilon}{\kappa}\log\left(\frac{y}{\alpha}\right). \tag{A5}$$

Furthermore, the Doppler shifted velocity $W = U_0 - c$ may be written in a number of forms. With y_c the critical height and

$$c = U_0(y_c) = 1 + \frac{\varepsilon}{\kappa}\log\left(\frac{y_c}{\alpha}\right). \tag{A6}$$

one finds

$$W = \frac{\varepsilon}{\kappa}\log\left(\frac{y}{y_c}\right). \tag{A7}$$

This form is appropriate for the so-called inner region to be defined shortly. Introducing $w = 1 - c$ and inverting Eq. (A6) the critical height can be written as

$$y_c = \alpha e^{-\frac{\kappa w}{\varepsilon}}, \tag{A8}$$

and elimination of y_c from (A7) gives an alternative form for W, namely

$$W = w + \frac{\varepsilon}{\kappa}\log(\frac{y}{\alpha}). \tag{A9}$$

This form will be used in the outer region.

Finally, we make a choice for the order of magnitude of the critical height y_c. It is assumed that

$$y_c = \mathscr{O}(\varepsilon) \tag{A10}$$

and therefore there are two regions/layers in this problem, an inner region of thickness $\mathscr{O}(\varepsilon)$, also called the critical layer, and an outer region of thickness $\mathscr{O}(1)$.

Next, the relevant solutions in these two layers are discussed. These will be matched afterwards providing an approximate solution to the problem (A1). In our expansion it will be assumed that y_0 is much smaller than ε and therefore effects of the order of y_0 will be ignored.

A.2 Critical layer

In the critical layer we introduce the new variable $\xi = y/\varepsilon$, then the relevant differential equation becomes

$$\left(W\frac{d^2}{d\xi^2} - W''\right)\Psi = \varepsilon^2 W\Psi \tag{A11}$$

where $W'' = -\varepsilon/(\kappa\xi^2)$ and $W = \frac{\varepsilon}{\kappa}\log(\xi/\xi_c)$. Note that $\xi_c = y_c/\varepsilon = \mathscr{O}(1)$.

Therefore, in the critical layer W'' and W are of the same order of magnitude and the RHS is a small correction. In order to obtain an approximate solution the Rayleigh equation is converted to an integral equation. We pose

$$\Psi = W\Phi$$

then

$$\frac{d}{d\xi}\left(W^2\frac{d}{d\xi}\Phi\right) = \varepsilon^2 W^2\Phi. \tag{A12}$$

Integrating twice, using the boundary condition $\Psi(y_0) = -c$, hence $\Phi(y_0) = 1$, one finds with $\xi_0 = y_0/\varepsilon$

$$\Phi = 1 + \varepsilon\beta \int_{\xi_0}^{\xi} \frac{d\eta}{W^2(\eta)} + \varepsilon^2 \int_{\xi_0}^{\xi} \frac{d\eta}{W^2(\eta)} \int_{\eta_0}^{\eta} d\zeta\, W^2(\zeta)\Phi(\zeta), \tag{A13}$$

where from the outset we have chosen the second term to be of order ε. The constant β will be determined by matching with the outer solution. In the present treatment we only need a solution up to first order in ε. Thus,

$$\Phi = 1 + \varepsilon\beta K(\xi),\; K(\xi) = \int_{\xi_0}^{\xi} \frac{d\eta}{W^2}, \tag{A14}$$

The integration contour in the integal defining the K-function is along the real axis, except near the critical height where the contour is deformed by indentation below the real axis (assuming that $U'_{0c} > 0$). In order to see this more clearly the integration variable is changed from η to W, hence

$$K(\xi) = \int_{W_0}^{W} \frac{dW}{W'W^2},$$

where $W_0 = -c$. The next step is to express W' in terms of W. With

$$W = \frac{\varepsilon}{\kappa} \log(\xi/\xi_c)$$

one may express ξ in terms of W, i.e.

$$\xi = \xi_c e^{\frac{\kappa W}{\varepsilon}},$$

and then

$$W' = \varepsilon/(\kappa\xi) = W'_c e^{-\frac{\kappa W}{\varepsilon}}.$$

As a consequence, with $x = \kappa W/\varepsilon$ and $x_0 = \kappa W_0/\varepsilon$, the function K may therefore be written as

$$K(\xi) = \frac{\kappa}{\varepsilon W'_c} \int_{x_0}^{x} dy \frac{e^y}{y^2}.$$

Partial integration then gives

$$K(\xi) = \frac{\kappa}{\varepsilon W'_c} \left(-\left.\frac{e^y}{y}\right|_{x_0}^{x} + \int_{x_0}^{x} dy \frac{e^y}{y} \right). \tag{A15}$$

The lower bound of the integral is in practice very large and negative. Therefore it may be replaced by $-\infty$. In that case the integral is connected to the Exponential Integral Ei defined as

$$Ei(x) = \int_{-\infty}^{x} dy \frac{e^y}{y} = \mathrm{P} \int_{-\infty}^{x} dy \frac{e^y}{y} + \pi i H(x),$$

with $H(x)$ the Heaviside function. Thus, for $x > 0$, i.e. above the critical height, the K-function becomes complex. There is therefore a phase jump above the critical height, a phenomenon that has been extensively observed by Hristov *et al.* (2003). Thus, finally, the K-function becomes to good approximation

$$K(\xi) = \frac{\kappa}{\varepsilon W'_c} \left(-\frac{e^x}{x} + Ei(x) \right) \tag{A16}$$

For matching purposes we need the inner solution for large ξ, hence $\xi >> \xi_c$. With

$$\mathrm{P}\int_{-\infty}^{x} \mathrm{d}y \frac{e^y}{y} = \frac{e^x}{x}\left(1+\frac{1}{x}+\frac{2}{x^2}+\dots\right)$$

one finds

$$\lim_{\xi\to\infty} K(\xi) = \frac{\xi}{W^2}\left(1+\frac{2\varepsilon}{\kappa W}\right)+\pi i\left(\frac{\kappa}{\varepsilon}\right)^2 \xi_c.$$

Therefore, for large ξ the inner solution becomes

$$\lim_{\xi\to\infty} \Psi_{inner} = W\left[1+\varepsilon\beta\left\{\frac{\xi}{W^2}\left(1+\frac{2\varepsilon}{\kappa W}\right)+\pi i\left(\frac{\kappa}{\varepsilon}\right)^2 \xi_c\right\}\right].$$

In order to match to the solution in the outer region Ψ_{inner} is written in terms of the independent variable $y=\varepsilon\xi$, while realizing that in the outer region the doppler shifted velocity is to first approximation a constant, i.e. $W = w+\frac{\varepsilon}{\kappa}\log(y/\alpha)$. As a result one finds

$$\Psi_{inner} \to w\left(1+\frac{\beta y}{w^2}\right)+\frac{\varepsilon}{\kappa}\left\{\frac{2\beta y}{w^2}+\log\left(\frac{y}{\alpha}\right)\left(1-\frac{\beta y}{w^2}\right)\right\}+ i\beta\Gamma\left[w+\frac{\varepsilon}{\kappa}\log\left(\frac{y}{\alpha}\right)\right] \tag{A17}$$

with

$$\Gamma = \pi\left(\frac{\kappa}{\varepsilon}\right)^2 y_c.$$

A.3 Outer layer.

In the outer layer we assume a different balance by taking the curvature term to be small, so that now

$$\left(W\frac{\mathrm{d}^2}{\mathrm{d}y^2}-1\right)\Psi = W''\Psi = \text{small}. \tag{A18}$$

In the outer layer we pose

$$\Psi = \Phi e^{-y},$$

then

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\mathrm{d}}{\mathrm{d}y}\Phi-2\Phi\right) = q\Phi,\ q=\frac{W''}{W},$$

hence, integrating once one has

$$\frac{\mathrm{d}}{\mathrm{d}y}\Phi-2\Phi = \int_y^\infty \mathrm{d}\eta\, q(\eta)\Phi(\eta)+const.$$

Another integration gives

$$\Phi = F+\frac{1}{2}\int_y^\infty \mathrm{d}\eta\left(1-e^{-2(\eta-y)}\right)q(\eta)\Phi(\eta).$$

In the outer layer $y = \mathscr{O}(1)$ and then q is found to be small as $q(\eta) = -\varepsilon/(\kappa w \eta^2)$. The above integral equation for Φ can then be solved by means of iteration and up to second order one finds

$$\Phi = F + \frac{1}{2} F \int_y^\infty \mathrm{d}\eta \left(1 - e^{-2(\eta - y)}\right) q(\eta).$$

Making use of the expression for q, the integral may be evaluated with the result

$$\Psi_{outer} = F e^{-y} \left(1 - \frac{\varepsilon}{\kappa w} e^{2y} E_1(2y)\right) + \ldots.$$

which is identical to van Duin and Janssen (1992). Here E_1 is another form of the exponential integral, i.e.

$$E_1(x) = \int_x^\infty \mathrm{d}t \frac{e^{-t}}{t}.$$

The unknown F will be determined by matching with the inner solution, hence we have to evaluate Ψ for small y. One finds

$$\lim_{y \to 0} \Psi_{outer} = F \left[1 - y + \frac{\varepsilon}{\kappa w} \left\{-2y + (1+y) \log\left(\frac{y}{a}\right)\right\}\right] \tag{A19}$$

with $\log(1/a) = \gamma + \log 2$. Here we used $E_1(2y) = 2y - \log(y/a)$ for $y \to 0$.

A.4 Matching the critical layer and the outer layer.

We match now (A17) and (A19). In our approach we have introduced one additional degree of freedom namely the height α of the scaling velocity V, see Eq. (A3), and this will be determined by higher order matching.

Let us start with lowest order matching of the inner and outer solution. Hence, we only consider terms that are of zeroth order in ε. Then

$$\Psi^{(0)}_{outer} = F(1-y), \text{and } \Psi^0_{inner} = w\left(1 + \frac{\beta y}{w^2} + i\beta\Gamma\right)$$

Matching the constant terms then gives

$$F = w(1 + i\beta\Gamma)$$

while matching the y-terms gives

$$\beta = -wF$$

Eliminating β from the above two equations then results in

$$F = \frac{w}{1 + i\hat{\Gamma}}, \tag{A20}$$

while

$$\beta = -\frac{w^2}{1 + i\hat{\Gamma}}, \tag{A21}$$

with $\hat{\Gamma} = w^2\Gamma = \pi(\kappa w/\varepsilon)^2 y_c$.

Let us now continue with first-order matching. In first order the outer solution gives the terms

$$\Psi^{(1)}_{outer} = \frac{F}{\kappa w}\{-2y + (1+y)(\log y - \log a)\}$$

while the first-order inner solution reads

$$\Psi^{(1)}_{inner} = \frac{F}{\kappa w}\{-2y + (1+y)(\log y - \log \alpha)\}.$$

First of all, note that the $(1+y)\log y$ terms match unconditionally. The $y-$terms match provided the following condition on a is satisfied:

$$\alpha = a = \frac{1}{2}e^{-\gamma}$$

As a result we obtain a solution which is correct to first order in ε.

A.5 Growth rate.

The growth rate γ_E of the energy E of the gravity waves is connected to the pressure perturbation p at the surface. From van Duin and Janssen (1992) one finds

$$\frac{\gamma_E}{\omega} = \frac{s}{c^2}\Im(p)$$

where the pressure perturbation in the critical layer follows from

$$p = \frac{1}{\varepsilon}\left\{W\frac{\mathrm{d}\Psi}{\mathrm{d}\xi} - \Psi\frac{\mathrm{d}W}{\mathrm{d}\xi}\right\}$$

Using $\Psi = W\Phi$ then

$$p = \frac{1}{\varepsilon}W^2\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}$$

Therefore, with (A14), one simply obtains

$$p = \beta = -\frac{w^2}{1+i\hat{\Gamma}}.$$

As a consequence, the growth rate becomes

$$\frac{\gamma_E}{\omega} = \frac{s}{c^2}\Im(p) = s\left(\frac{w}{c}\right)^2\frac{\hat{\Gamma}}{1+\hat{\Gamma}^2}$$

Recall that $\hat{\Gamma} = w^2\Gamma = \pi(\kappa w/\varepsilon)^2 y_c$. Here, velocities have been scaled with V. In terms of friction velocity scaling one finds for the growth rate

$$\frac{\gamma_E}{\omega} = s\beta_M\left(\frac{u_*}{c}\right)^2 \tag{A22}$$

where

$$\beta_M = \left(\frac{w}{\varepsilon}\right)^2\frac{\hat{\Gamma}}{1+\hat{\Gamma}^2}$$

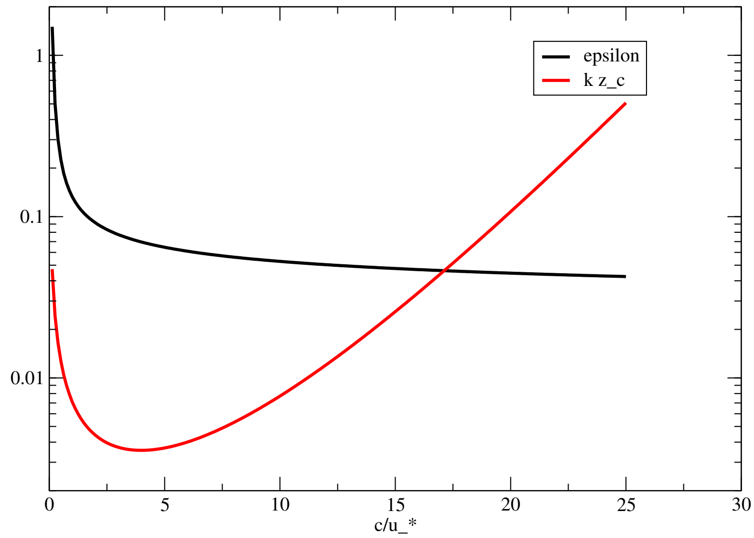

Figure 6: The small parameter ε and kz_c as function of the dimensionless phase speed c/u_*. Both parameters should be small for a valid expansion.The Charnock parameter equals 0.0144.

Now w/ε is connected to the critical height according to $w/\varepsilon = -\frac{1}{\kappa}\log(y_c/\alpha)$, thus

$$\hat{\Gamma} = \pi y_c \log^2(y_c/\alpha)$$

while

$$\beta_M = \frac{\pi}{\kappa^2}\frac{y_c \log^4(y_c/\alpha)}{1+\pi^2 y_c^2 \log^4(y_c/\alpha)}.$$

In practice, it turns out that the factor in the denominator is relatively small so that in good approximation

$$\beta_M \approx \frac{\pi}{\kappa^2} y_c \log^4(y_c/\alpha). \tag{A23}$$

This result, including the additional Z-factor (cf. Eq. (34), has been plotted in Fig. 2. While for the short waves a reasonable agreement with the numerical results is found, it is clear that the above approximation fails for the long waves. The reason for this is that for the long waves there is no real distinction between the critical layer and the outer layer as kz_c becomes order 1, as illustrated in Fig. 6. In addition Fig. 6 shows the dependence of the small parameter ε on dimensionless phase speed c/u_*. Clearly for the very short waves ε becomes of order 1, therefore both for very short waves and for very long waves the asymptotic solution is strictly speaking not valid.

Finally, it is of interest to determine the pressure profile in the outer region, because this is the region which is usually accesible to observations. In the outer layer one has

$$p_{outer} = W\frac{d\Psi}{dy} - \Psi\frac{dW}{dy}$$

and using the outer solution Ψ_{outer} one obtains

$$p_{outer} = -FWe^{-y}\left\{1 + \frac{\varepsilon}{\kappa w}e^{2y}E_1(2y)\right\} + \mathcal{O}(\varepsilon^2)$$

where $W = w + \frac{\varepsilon}{\kappa}\log(y/\alpha)$ and $F = w/(1+i\hat{\Gamma})$. In order to check that we have followed a proper matching procedure one would expect that the pressure is continuous and that the limit of small height of the outer pressure should be equal to the critical layer pressure. This is indeed the case as

$$p_{outer} = -FWe^{-y} + \mathcal{O}(\varepsilon) = -\frac{w^2}{1+i\hat{\Gamma}}$$

which agrees with the critical layer pressure result $p = \beta$.

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