A Unified Framework for Discrete Integrations of Compressible and Soundproof PDEs of Atmospheric Dynamics

Piotr Smolarkiewicz, Christian Kühnlein, Nils Wedi

European Research Council
Established by the European Commission
Supporting top researchers from anywhere in the world

- A common approach for consistent integrations of soundproof and compressible nonhydrostatic PDEs for all-scale atmospheric dynamics.

- Generalisation of proven soundproof numerics (anelastic, pseudo-incompressible) to low speed compressible solvers (acoustic, semi-implicit, flux-form Eulerian and semi-Lagrangian)

- Extension of variational Krylov solvers for generalised Poisson problem to corresponding generalised Helmholtz problems of two kinds.

Anelastic equations of Lipps and Hemler (1982, 1990) :

$$\frac{d\mathbf{u}}{dt} = -\nabla\phi - \mathbf{g}\frac{\theta - \theta_b}{\theta_b} - \mathbf{f}\times\mathbf{u} \ ,$$

$$\frac{d\theta}{dt} = 0$$

$$\nabla\cdot(\rho_b\mathbf{u}) = 0 \ .$$

$$ds = c_p d\ln\theta \qquad \mathbf{f} \equiv 2\mathbf{\Omega}$$

$$S = d\ln\theta_b/dz = N^2/g \geq 0$$

$$\phi \equiv c_p\theta_b(\pi - \pi_b) \text{ with } \pi \equiv (p/p_0)^{R_d/c_p}$$

$$\rho^{-1}\nabla p = c_p\theta\nabla\pi; \text{ and } \pi = T/\theta$$

$$\nabla = (\partial_x, \partial_y, \partial_z) \qquad d/dt = \partial/\partial t + \mathbf{v}\cdot\nabla$$

$$0 = -\nabla\phi_e - \mathbf{g}\frac{\theta_e - \theta_b}{\theta_b} - \mathbf{f}\times\mathbf{u}_e$$

$$\frac{d\mathbf{u}}{dt} = -\nabla\phi' - \mathbf{g}\frac{\theta'}{\theta_b} - \mathbf{f}\times\mathbf{u}' \ ,$$

$$\frac{d\theta'}{dt} = -\mathbf{u}\cdot\nabla\theta_e \ ,$$

$$\nabla\cdot(\rho_b\mathbf{u}) = 0 \ ,$$

Pseudoincompressible equations of Durran (JAS 1989, JFM 2008)

$$\frac{d\mathbf{u}}{dt} = -c_p\theta\nabla\pi' - \mathbf{g}\frac{\theta'}{\theta_e} - \mathbf{f}\times\left(\mathbf{u} - \frac{\theta}{\theta_e}\mathbf{u}_e\right) ,$$

$$\frac{d\theta'}{dt} = -\mathbf{u}\cdot\nabla\theta_e ,$$

$$\nabla\cdot\left(\rho_b\frac{\theta_b}{\theta_0}\mathbf{u}\right) = 0 ,$$

$$-c_p\theta\nabla\pi' = -\nabla\phi' + \phi'\nabla\ln\theta$$

$$0 = -c_p\theta_e\nabla(\pi_e - \pi_b) - \mathbf{g}\frac{\theta_e - \theta_b}{\theta_b} - \mathbf{f}\times\mathbf{u}_e$$

Compressible Euler equations (e.g., Dutton 1986) :

$$\frac{d\mathbf{u}}{dt} = -c_p\theta\nabla\pi' - \mathbf{g}\frac{\theta'}{\theta_e} - \mathbf{f}\times\left(\mathbf{u} - \frac{\theta}{\theta_e}\mathbf{u}_e\right) ,$$

$$\frac{d\theta'}{dt} = -\mathbf{u}\cdot\nabla\theta_e ,$$

$$\frac{d\rho}{dt} = -\rho\nabla\cdot\mathbf{u} ,$$

$$\pi = \left(\frac{R_d}{p_0}\rho\theta\right)^{R_d/c_v} ,$$

Combined symbolic equations:

$$\frac{d\mathbf{u}}{dt} = -\Theta\nabla\varphi - \mathbf{g}\Upsilon_B\frac{\theta'}{\theta_b} - \mathbf{f}\times(\mathbf{u}-\Upsilon_C\mathbf{u}_e) \ ,$$

$$\frac{d\theta'}{dt} = -\mathbf{u}\cdot\nabla\theta_e \ ,$$

$$\frac{d\varrho}{dt} = -\varrho\nabla\cdot\mathbf{u} \ .$$

$$\Theta := \left[1,\ \frac{\theta(\mathbf{x},t)}{\theta_0},\ \frac{\theta(\mathbf{x},t)}{\theta_0}\right] \ ,$$

$$\Upsilon_B := \left[1,\ \frac{\theta_b(z)}{\theta_e(\mathbf{x})},\ \frac{\theta_b(z)}{\theta_e(\mathbf{x})}\right] \ ,$$

$$\Upsilon_C := \left[1,\ \frac{\theta(\mathbf{x},t)}{\theta_e(\mathbf{x})},\ \frac{\theta(\mathbf{x},t))}{\theta_e(\mathbf{x})}\right] \ ,$$

$$\varrho := \left[\rho_b(z),\ \rho_b\frac{\theta_b(z)}{\theta_0},\ \rho(\mathbf{x},t)\right] \ ,$$

$$\varphi := \left[c_p\theta_b\pi',\ c_p\theta_0\pi',\ c_p\theta_0\pi'\right] \ ,$$

$$\varphi = c_p\theta_0\left[\left(\frac{R_d}{p_0}\varrho\theta\right)^{R_d/c_v} - \pi_e\right]$$

Combined equations, conservation form:

$$\frac{\partial \varrho \mathbf{u}}{\partial t} + \nabla \cdot (\varrho \mathbf{u} \otimes \mathbf{u}) = \varrho \boldsymbol{R}^{\mathbf{u}} \ ,$$

$$\frac{\partial \varrho \theta'}{\partial t} + \nabla \cdot (\varrho \theta') = \varrho R^{\theta} \ ,$$

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \mathbf{u}) = 0 \ ,$$

$$d\psi/dt = R$$

$$\partial \varrho \psi / \partial t + \nabla \cdot (\varrho \mathbf{u} \psi) = \varrho R$$

specific vs. density variables

Riemannian connection:

$$\frac{\partial \mathcal{G} \varrho \psi}{\partial t} + \nabla \cdot (\mathcal{G} \varrho \mathbf{v} \psi) = \mathcal{G} \varrho R$$

$$\frac{\partial \mathcal{G} \varrho}{\partial t} + \nabla \cdot (\mathcal{G} \varrho \mathbf{v}) = 0$$

$\mathbf{v} = \dot{\mathbf{x}}$ not necessarily equal to $\mathbf{u}$

$\mathcal{G}(\mathbf{x}, t)$ denotes the Jacobian

$\mathcal{G}^2$ is the determinant of the metric tensor

Integration schemes ➔ ➔ ➔

Non-oscillatory forward-in-time differencing for fluids: (archetype problem "AP")

$$\frac{\partial G\Psi}{\partial t} + \nabla \cdot (\mathbf{V}\Psi) = G\mathcal{R} \quad \rightarrow \quad \frac{G^{n+1}\Psi^{n+1} - G^n\Psi^n}{\delta t} + \nabla \cdot (\mathbf{V}^{n+1/2}\Psi^n) = \left(G\mathcal{R}\right)^{n+1/2}$$

→ *"modified equation":*

$$\begin{aligned}\frac{\partial G\Psi}{\partial t} &+ \nabla \cdot (\mathbf{V}\Psi) = GR \\ &- \nabla \cdot \left[\frac{\delta t}{2} G^{-1}\mathbf{V}(\mathbf{V}\cdot\nabla\Psi) + \frac{\delta t}{2} G^{-1}\left(\frac{\partial G}{\partial t} + \nabla\cdot\mathbf{V}\right)\mathbf{V}\Psi\right] \\ &+ \nabla \cdot \left(\frac{\delta t}{2}\mathbf{V}\mathcal{R}\right) + \mathcal{O}(\delta t^2)\ .\end{aligned}$$

For the homogeneous AP problem, compensating for all first-order errors leads to:

$$\begin{aligned}\Psi_{\mathrm{i}}^{n+1} &= \frac{G_{\mathrm{i}}^{n}}{G_{\mathrm{i}}^{n+1}}\left(\Psi_{\mathrm{i}}^{n} - \frac{\delta t}{G_{\mathrm{i}}^{n}}\nabla\cdot\overline{(\mathbf{V}\Psi)}^{n+1/2}d\tau\right) + \mathcal{O}(\delta t)^3 \\ &\equiv \mathcal{A}_{\mathrm{i}}\left(\Psi^n, \mathbf{V}^{n+1/2}, G^n, G^{n+1}\right)\ ,\end{aligned}$$

e.g.; MPDATA,
Kühnlein et al. 2012

Given availability of an 2nd order NFT algorithm for the homogeneous problem a 2nd order-accurate solution for an inhomogeneous problem is:

$$\Psi_{\mathbf{i}}^{n+1} = \mathcal{A}_{\mathbf{i}}\left(\widetilde{\Psi}^{n}, \mathbf{V}^{n+1/2}, G^{n}, G^{n+1}\right) + 0.5\delta t \mathcal{R}_{\mathbf{i}}^{n+1} ,$$

where

$$\widetilde{\Psi}^{n} \equiv \Psi^{n} + 0.5\delta t \mathcal{R}^{n} .$$

Dual interpretation of the archetype PDE $$\frac{\partial G\Psi}{\partial t} + \nabla \cdot (\mathbf{V}\Psi) = G\mathcal{R}$$

gas dynamics:

$$\Psi \equiv \varrho\psi, \quad G \equiv \mathcal{G} \text{ and } \mathcal{R} \equiv \varrho R$$

$$\mathbf{V} \equiv \mathcal{G}\mathbf{v}$$

soundproof models:

$$G \equiv \mathcal{G}\varrho, \quad \Psi \equiv \psi \text{ and } \mathcal{R} \equiv R$$

$$\mathbf{V} \equiv \mathcal{G}\varrho\mathbf{v}$$

$$\mathbf{V} \equiv G\mathbf{v}$$

unified framework, element 1

RE compressible Euler PDEs

$$\frac{\partial \mathcal{G}\varrho}{\partial t} + \nabla \cdot (\mathcal{G}\varrho\mathbf{v}) = 0 \ ,$$

$$\frac{\partial \mathcal{G}\varrho\theta'}{\partial t} + \nabla \cdot (\mathcal{G}\varrho\theta'\mathbf{v}) = -\mathcal{G}\varrho\widetilde{\mathbf{G}}^T\mathbf{u} \cdot \nabla\theta_e \ ,$$

$$\frac{\partial \mathcal{G}\varrho\mathbf{u}}{\partial t} + \nabla \cdot (\mathcal{G}\varrho\mathbf{v} \otimes \mathbf{u}) = -\mathcal{G}\varrho\left(\Theta\widetilde{\mathbf{G}}\nabla\varphi + \mathbf{g}\Upsilon_B\frac{\theta'}{\theta_b} + \mathbf{f} \times (\mathbf{u} - \Upsilon_C\mathbf{u}_e) - \mathcal{M}'(\mathbf{u}, \mathbf{u}, \Upsilon_C)\right)$$

semi-implicit ``acoustic" scheme:

$$\varrho_{\mathrm{i}}^{n+1} = \mathcal{A}_{\mathrm{i}}\left(\varrho^n, (\mathcal{G}\mathbf{v})^{n+1/2}, \mathcal{G}, \mathcal{G}\right) \quad \Longrightarrow \quad \mathbf{V}^{n+1/2} = \overline{(\mathcal{G}\varrho\mathbf{v})}^{n+1/2}$$

$$\widehat{\theta}'_{\mathrm{i}} = \mathcal{A}_{\mathrm{i}}\left(\widetilde{\theta}', \mathbf{V}^{n+1/2}, \varrho^{*\,n}, \varrho^{*\,n+1}\right)$$

$$\widehat{\mathbf{u}}_{\mathrm{i}} = \mathcal{A}_{\mathrm{i}}\left(\widetilde{\mathbf{u}}, \mathbf{V}^{n+1/2}, \varrho^{*\,n}, \varrho^{*\,n+1}\right)$$

$$\varrho^{*\,n} := \mathcal{G}\varrho^n \text{ and } \varrho^{*\,n+1} := \mathcal{G}\varrho^{n+1}$$

$$\nu = 1, .., N_\nu$$

$$\theta'|_{\mathrm{i}}^{\nu} = \widehat{\theta}'_{\mathrm{i}} - 0.5\delta t\left(\widetilde{\mathbf{G}}^T\mathbf{u}^\nu \cdot \nabla\theta_e\right)_{\mathrm{i}}$$

$$\mathbf{u}_{\mathrm{i}}^{\nu} = \widehat{\mathbf{u}}_{\mathrm{i}} - 0.5\delta t\left(\Theta^{\nu-1}\widetilde{\mathbf{G}}\nabla\varphi^\nu + \mathbf{g}\Upsilon_B\frac{\theta'^{\nu}}{\theta_b}\right)_{\mathrm{i}} - 0.5\delta t\left(\mathbf{f} \times (\mathbf{u}^\nu - \Upsilon_C^{\nu-1}\mathbf{u}_e) - \mathcal{M}'(\mathbf{u}, \mathbf{u}, \Upsilon_C)^{\nu-1}\right)_{\mathrm{i}}$$

$$\varphi_{\mathrm{i}}^{\nu} = c_p\theta_0\left[\left(\frac{R_d}{p_0}\varrho^{n+1}\theta^{\nu-1}\right)^{R_d/c_v} - \pi_e\right]_{\mathrm{i}}$$

$$\theta_{\mathrm{i}}^{\nu} = \left(\widehat{\theta}' - 0.5\delta t\widetilde{\mathbf{G}}^T\mathbf{u}^\nu \cdot \nabla\theta_e + \theta_e\right)_{\mathrm{i}} \qquad \theta_{\mathrm{i}}^{0} = \mathcal{A}_{\mathrm{i}}\left(\theta^n, \mathbf{V}^{n+1/2}, \varrho^{*\,n}, \varrho^{*\,n+1}\right)$$

$$\overbrace{\mathbf{u}^{\nu} + 0.5\delta t\mathbf{f} \times \mathbf{u}^{\nu} - (0.5\delta t)^2 \mathbf{g}\Upsilon_B \frac{1}{\theta_b} \widetilde{\mathbf{G}}^T \mathbf{u}^{\nu} \cdot \nabla\theta_e}^{\mathbf{L}\mathbf{u}^{\nu}} =$$

$$\hat{\mathbf{u}} - 0.5\delta t \left(\mathbf{g}\Upsilon_B \frac{\widehat{\theta'}}{\theta_b} - \mathbf{f} \times \Upsilon_C^{\nu-1} \mathbf{u}_e - \mathcal{M}'(\mathbf{u}, \mathbf{u}, \Upsilon_C)^{\nu-1} \right)$$

$$-0.5\delta t \Theta^{\nu-1} \widetilde{\mathbf{G}} \nabla\varphi^{\nu} \equiv \boxed{\hat{\hat{\mathbf{u}}} - 0.5\delta t \Theta^{\nu-1} \widetilde{\mathbf{G}} \nabla\varphi^{\nu}}$$

$$\mathbf{u}^{\nu} = \check{\check{\mathbf{u}}} - \mathbf{C}\nabla\varphi^{\nu} \quad \text{where } \check{\check{\mathbf{u}}} = \mathbf{L}^{-1}\hat{\hat{\mathbf{u}}} \text{ and } \mathbf{C} = \mathbf{L}^{-1} 0.5\delta t \Theta^{\nu-1} \widetilde{\mathbf{G}}$$

unified framework, element 2: Helmholtz solvers for compressible models: :

0) Poisson problem in soundproof models

$$\nabla \cdot (\varrho^* \mathbf{v}) = 0$$

Because $\mathbf{v} = \widetilde{\mathbf{G}}^T \mathbf{u}$, acting with $\widetilde{\mathbf{G}}^T$ on both sides of $\mathbf{u}^{\nu} = \check{\check{\mathbf{u}}} - \mathbf{C}\nabla\varphi^{\nu}$ and …

$$0 = -\frac{\delta t}{\varrho^*} \nabla \cdot (\varrho^* \mathbf{v}^{\nu}) = -\frac{\delta t}{\varrho^*} \nabla \cdot \left[\varrho^* \left(\check{\check{\mathbf{v}}} - \widetilde{\mathbf{G}}^T \mathbf{C} \nabla\varphi^{\nu} \right) \right]$$

A diagonally preconditioned Poisson problem for pressure perturbation .

1) Helmholtz problem for compressible NFT model; "first kind"

Combine the gas law and mass continuity equation into the inhomogeneous AP

$$\frac{d\pi}{dt} = -\gamma\pi\nabla\cdot\mathbf{u} \implies \frac{\partial\rho\pi}{\partial t} + \nabla\cdot(\rho\pi\mathbf{u}) = -\gamma\rho\pi\nabla\cdot\mathbf{u} \quad \text{where } \gamma \equiv R_d/c_v$$

$$\frac{\partial\varrho^*\pi}{\partial t} + \nabla\cdot(\varrho^*\mathbf{v}\pi) = -\gamma\varrho^*\pi\frac{1}{\mathcal{G}}\nabla\cdot(\mathcal{G}\mathbf{v})$$

$$\pi^{n+1} = \widehat{\pi} - \delta t\gamma\pi^{n+1}\frac{1}{\mathcal{G}}\nabla\cdot\left(\mathcal{G}\mathbf{v}^{n+1}\right) + \mathcal{O}(\delta t^2)$$

$$\widehat{\pi} = \mathcal{A}\left(\pi^n, \mathbf{V}^{n+1/2}, \varrho^{*\,n}, \varrho^{*\,n+1}\right)$$

Poisson operator

$$0 = -\frac{\delta t}{\mathcal{G}}\nabla\cdot\left[\mathcal{G}\left(\check{\check{\mathbf{v}}} - \widetilde{\mathbf{G}}^T\mathbf{C}\nabla\varphi^\nu\right)\right] - \beta(\varphi^\nu - \varphi^\dagger)$$

Helmholtz term

$$\beta \equiv [\gamma(\varphi^{\nu-1} + c_p\theta_0\pi_e)i]^{-1},\ \varphi^\dagger \equiv c_p\theta_0(\widehat{\pi} - \pi_e)$$

dt 0.5 dt_soundproof, $(\nabla z)\cdot\mathbf{v}^{n+1/2}\partial_z\pi_e \sim g/\theta_e$ ➔ *Helmholtz problem of the "second kind"*

2) Helmholtz problem for compressible NFT model; "second kind"

formulate up front the inhomogeneous AP for pressure perturbation (rather than for full pressure to then derive the Helmholtz problem for the pressure perturbation) in the spirit of the equation;

$$\frac{d\pi'}{dt} = -\gamma\pi\nabla\cdot\mathbf{u} - \mathbf{u}\cdot\nabla\pi_e \;\Rightarrow\; \frac{\partial\rho\pi'}{\partial t} + \nabla\cdot(\rho\pi'\mathbf{u}) = -\gamma\rho\pi\nabla\cdot\mathbf{u} - \rho\mathbf{u}\cdot\nabla\pi_e$$

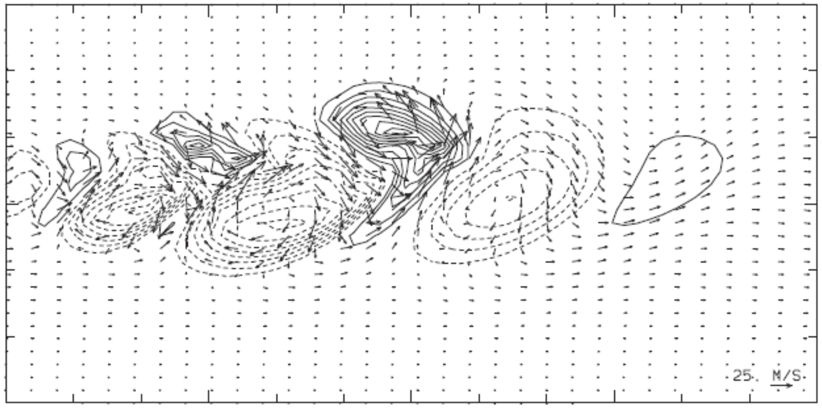
$$\frac{\partial\varrho^*\pi'}{\partial t} + \nabla\cdot(\varrho^*\mathbf{v}\pi') = -\left[\gamma\varrho^*\pi\frac{1}{\mathcal{G}}\nabla\cdot(\mathcal{G}\mathbf{v}) + \nabla\cdot(\varrho^*\mathbf{v}\pi_e) - \pi_e\nabla\cdot(\varrho^*\mathbf{v})\right]$$

$$0 = -\,\delta t\left[\frac{1}{\mathcal{G}}\nabla\cdot(\mathcal{G}\mathbf{v}) + \frac{1}{\gamma}\frac{\pi_e}{\pi}\frac{1}{\varrho^*\pi_e}\nabla\cdot(\varrho^*\pi_e\mathbf{v}) - \frac{1}{\gamma}\frac{\pi_e}{\pi}\frac{1}{\varrho^*}\nabla\cdot(\varrho^*\mathbf{v})\right] - \beta(\varphi - \widehat{\varphi})$$

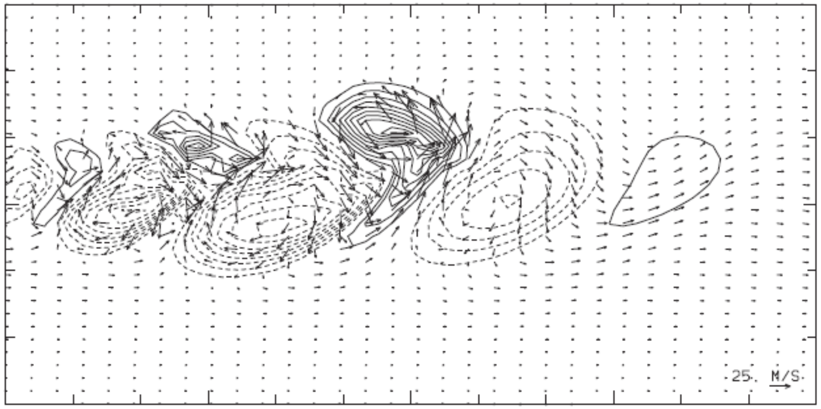
Results → → →

1.Global baroclinic instability; Prusa & Gutowski (2010, ECCOMAS paper #1453, adaptation of Jablonowski & Williamson (2006, QJR) ➔ Smolarkiewicz (2011, ECMWF)

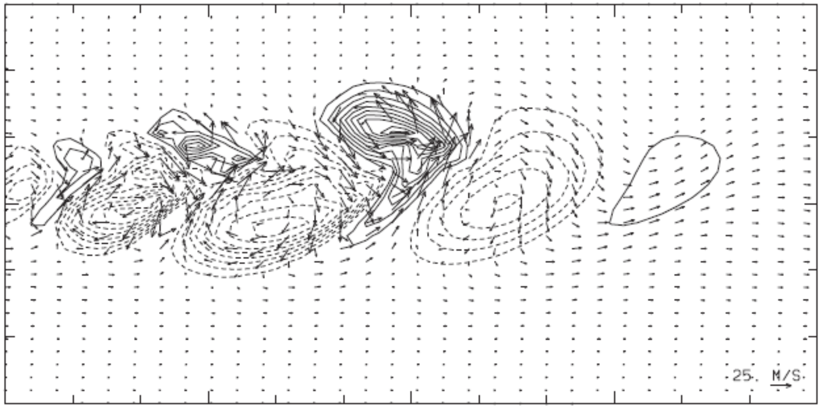
8 days, surface θ’,
128x64x48 lon-lat grid,
128 PE of Power7 IBM



CPI2, 2880 dt=300 s,
wallclock time=2.0 mns,
wlt/dt=0.041 s

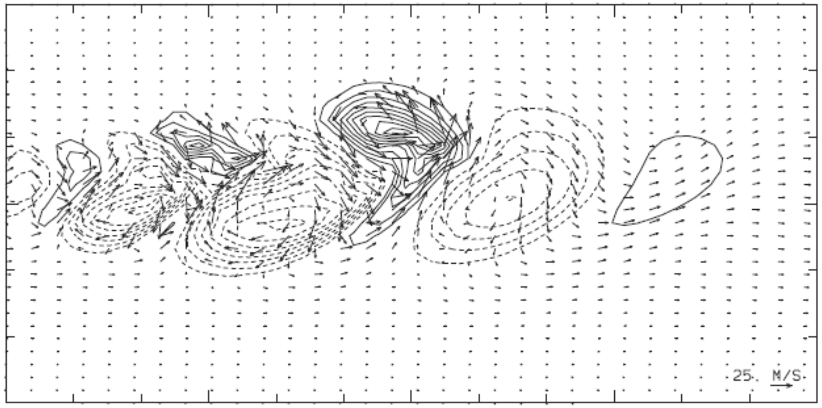


CPI1, 5760 dt=150 s,
wallclock time=3.7 mns,
wlt/dt=0.039 s

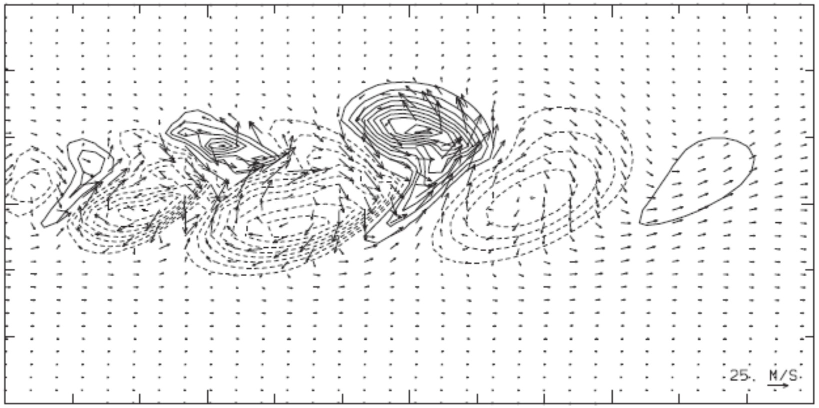


CPEX, 432000 dt=2 s,
wallclock time=178.9 mns,
wlt/dt=0.025 s

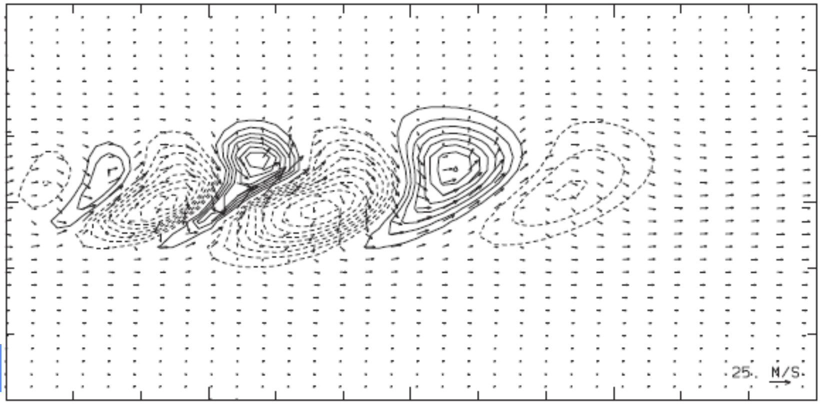
1.Global baroclinic instability; Prusa & Gutowski (2010, ECCOMAS paper #1453, adaptation of Jablonowski & Williamson (2006, QJR) ➔ Smolarkiewicz (2011, ECMWF)



CPI2, 2880 dt=300 s,
wallclock time=2.0 mns,
wlt/dt=0.041 s

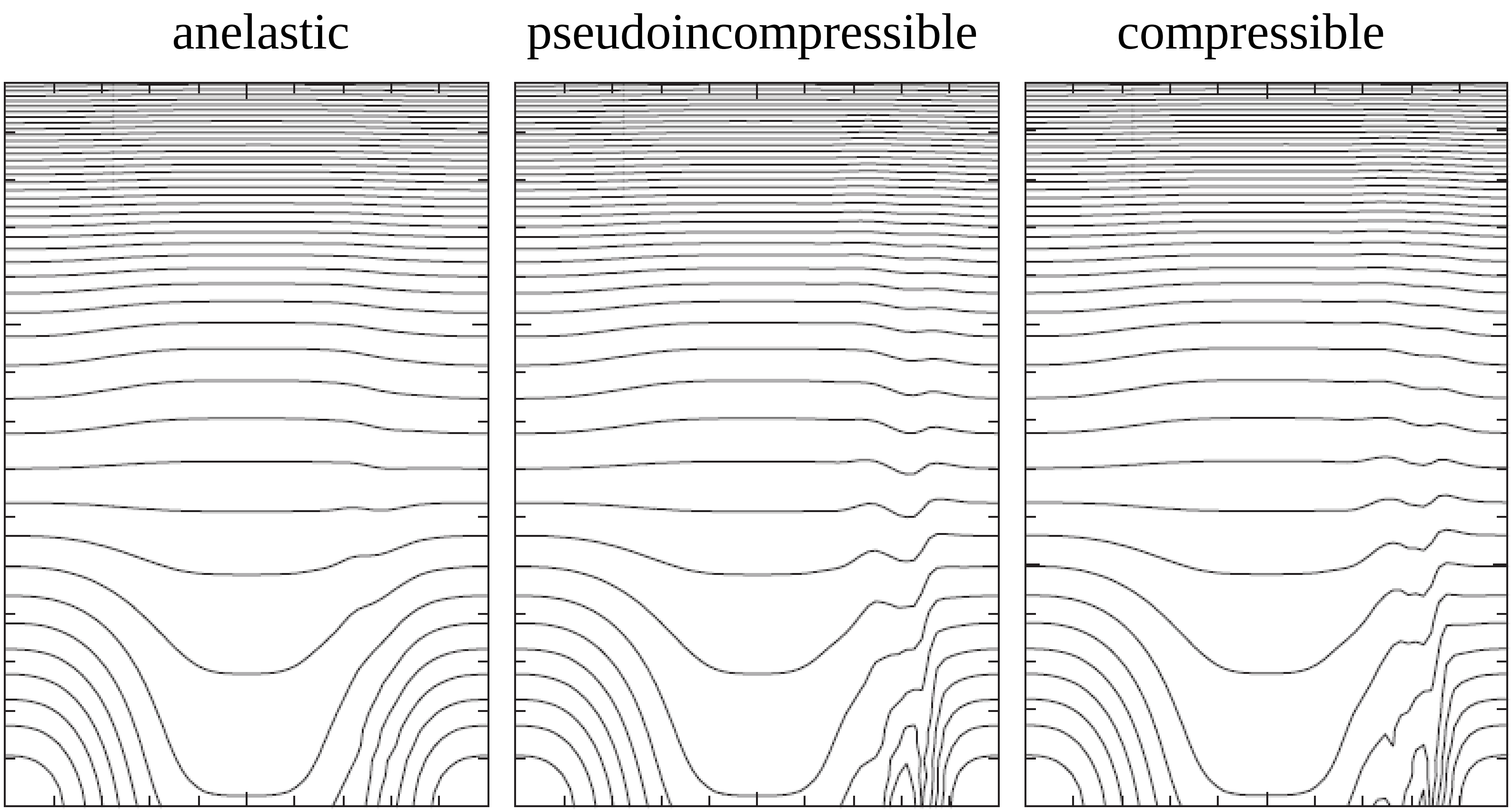


PSI, 2880 dt=300 s,
wallclock time=2.3 mns,
wlt/dt=0.048 s



ANL, 2880 dt=300 s,
wallclock time=2.1 mns,
wlt/dt=0.044 s

The role of baroclinicity

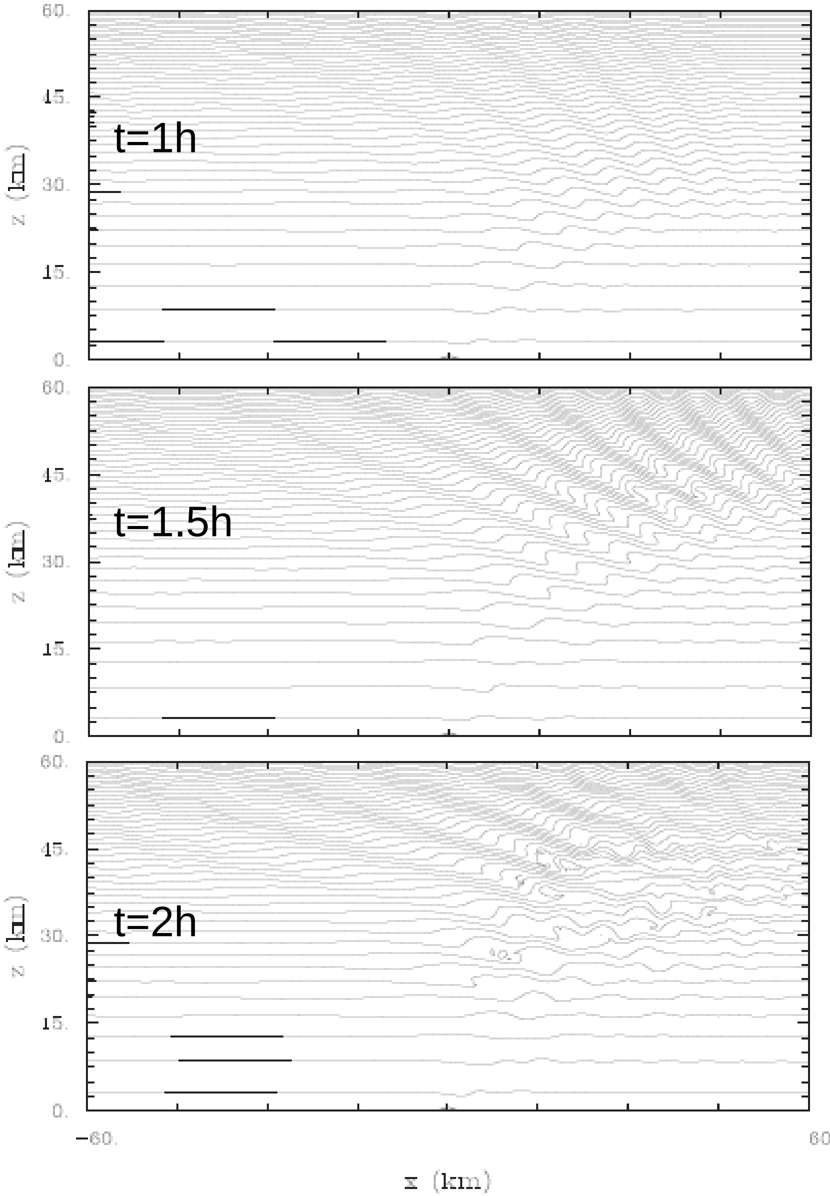


Non-Boussinesq amplification and breaking of deep stratospheric gravity wave

isothermal reference profiles ; $H_\theta = 3.5\, H_\rho$

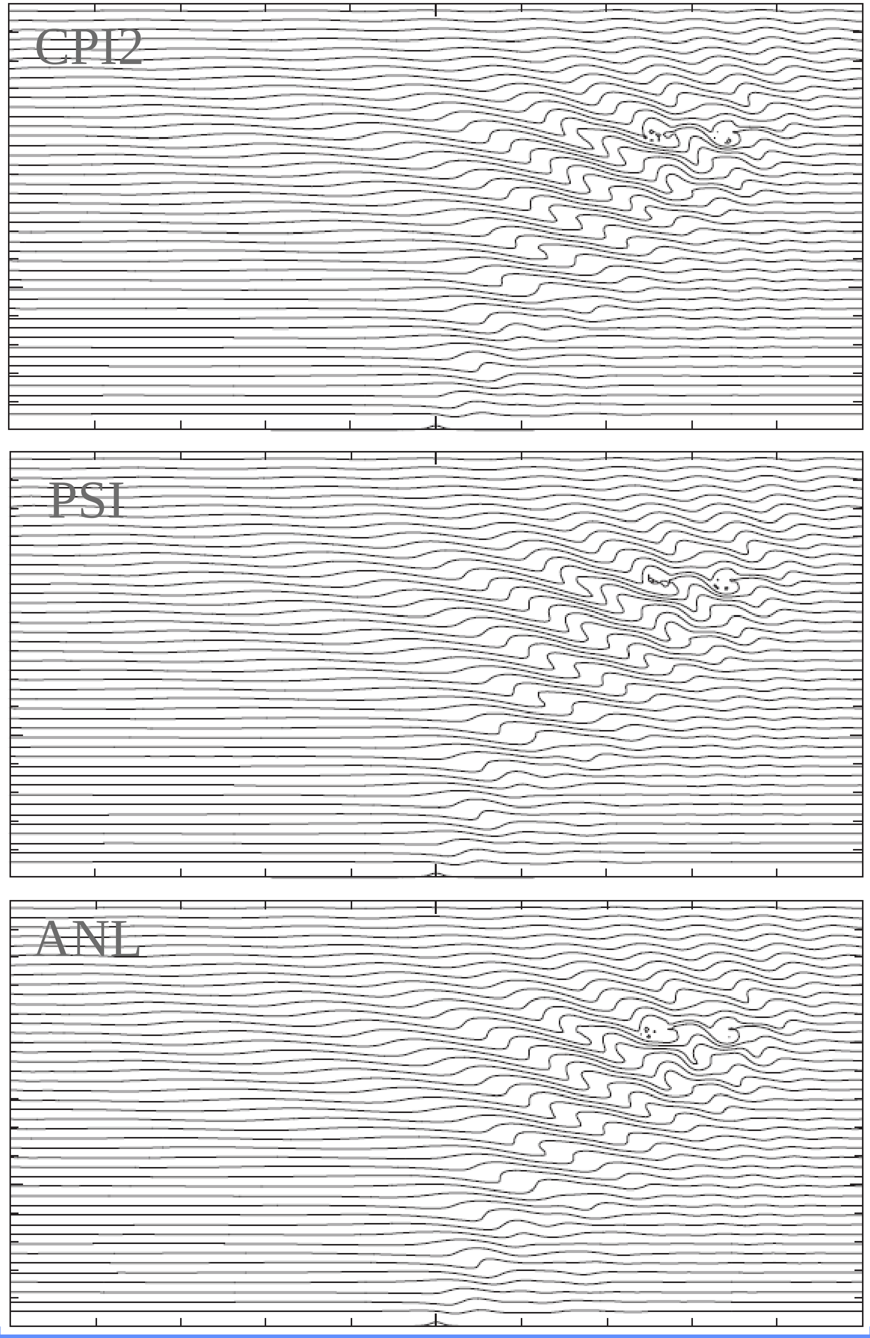
$NL/U_o \approx 1$, $Fr \approx 1.6$;
$\lambda_o = 2\pi$ km $<\approx H_\rho \Rightarrow$
$A(H/2)=10h_o = \lambda_o$

anelastic conservative reference solution in terrain-following coordinates ➔



Prusa et al., *JAS* 1996;
Smolarkiewicz & Margolin, *Atmos. Ocean*,1997;
Klein, *Ann. Rev. Fluid Dyn.*, 2010
Smolarkiewicz & Szmelter, Acta Geophysica, 2011

1.5h, surface lnθ,
320x160 Gal-Chen grid,
domain 120 km x 60 km
``soundproof'' dt=5 s
``acoustic'' dt=0.5 s
320 PE of Power7 IBM



ECMWF

European Research Council
Established by the European Commission
Supporting top researchers from anywhere in the world

Principal results

- Soundproof and compressible models are elements of a more general theoretical-numerical framework underlying non-oscillatory forward-in-time (NFT) approach
- The respective PDEs are integrated using essentially the same numerics
- The resulting compressible solvers are available in compatible flux-form Eulerian and semi-Lagrangian variants
- The flux-form solvers readily extend to unstructured-meshes

- The acoustic and large time step solutions for a synoptic scale problem of global baroclinic instability closely match each other
- Pseudoincompressible solutions for the synoptic scale problem closely approximate compressible results
- All solutions match each other for a mesoscale problem of deep gravity wave

The unified framework is a tool for blending models; RE Arakawa-Konor 2009

The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2012/ERC Grant agreement no. 320375)

What is A operator? MPDATA
(*multidimensional positive definite advection transport algorithm*)

$$\frac{\partial \phi}{\partial t} = -\nabla \bullet (\mathbf{V}\phi) \;, \quad \Rightarrow \quad \phi_i^{n+1} = \phi_i^n - \frac{\delta t}{\mathcal{V}_i} \sum_{j=1}^{l(i)} F_j^{\perp} S_j$$

$$F_j^{\perp}(\phi_i, \phi_j, V_j^{\perp}) = [V_j^{\perp}]^{+}\phi_i + [V_j^{\perp}]^{-}\phi_j \;, \quad [V]^{+} \equiv 0.5(V + |V|) \;, \quad [V]^{-} \equiv 0.5(V - |V|) \;,$$

$$\phi_i^{(k)} = \phi_i^{(k-1)} - \frac{\delta t}{\mathcal{V}_i} \sum_{j=1}^{l(i)} F_j^{\perp}\left(\phi_i^{(k-1)}, \phi_j^{(k-1)}, V_j^{\perp,(k)}\right) S_j$$

with $k = 1, .., IORD$ such that

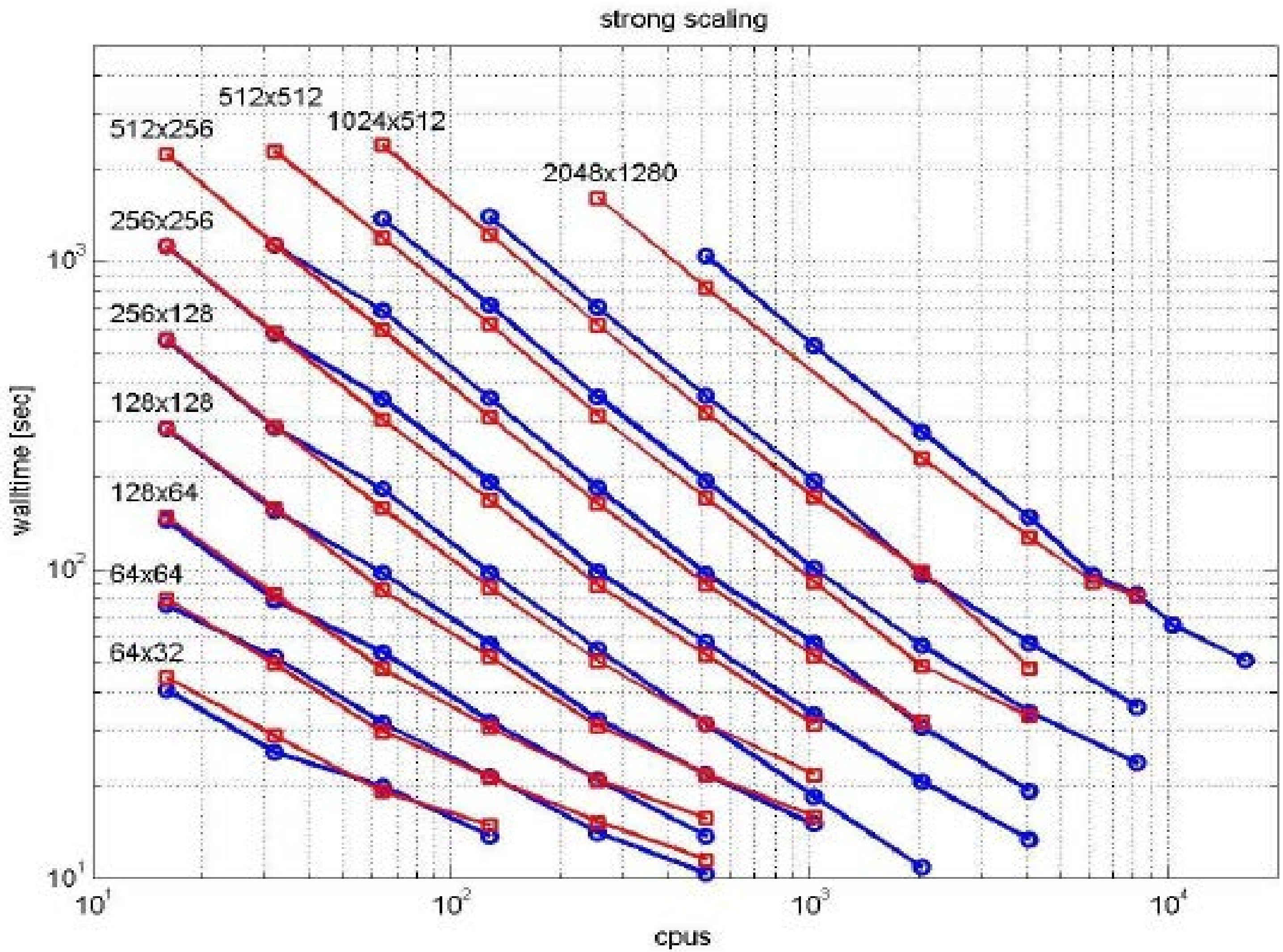
$$\phi^{(0)} \equiv \phi^{n} \quad ; \quad \phi^{(IORD)} \equiv \phi^{n+1}$$

$$V^{\perp,(k+1)} = V^{\perp}\left(\mathbf{V}^{(k)}, \phi^{(k)}, \nabla\phi^{(k)}\right) \quad ; \quad V_j^{\perp,(1)} \equiv V^{\perp}\big|_j^{n+1/2}$$

$$V^{\perp}\Big|_{s_j}^{(k+1)} = \Bigg\{ 0.5|V^{\perp}| \left(\frac{1}{|\phi|}\frac{\partial|\phi|}{\partial r}\right)(r_j - r_i) - 0.5V^{\perp}\left(\frac{1}{|\phi|}\frac{\partial|\phi|}{\partial r}\right)(r_i - 2r_{s_j} + r_j) - 0.5\delta t V^{\perp}\left(\mathbf{V} \bullet \frac{1}{|\phi|}\nabla|\phi|\right) - 0.5\delta t V^{\perp}(\nabla \bullet \mathbf{V}) \Bigg\}\Bigg|_{s_j}^{(k)}$$

In return for complexity MPDATA offers: nonlinear stability, independence on spatial discretization, and scalability ➔

Figure 2: Strong scalability, the total CPU time versus the number of processors for MPI implementation of EULAG on NCAR's IBM Blue Gene/L and IBM Blue Gene/W systems. The horizontal mesh resolution is indicated, and the vertical grid size is fixed at 41.



ECMWF

Example #1: Power of ILES; L2 integrability

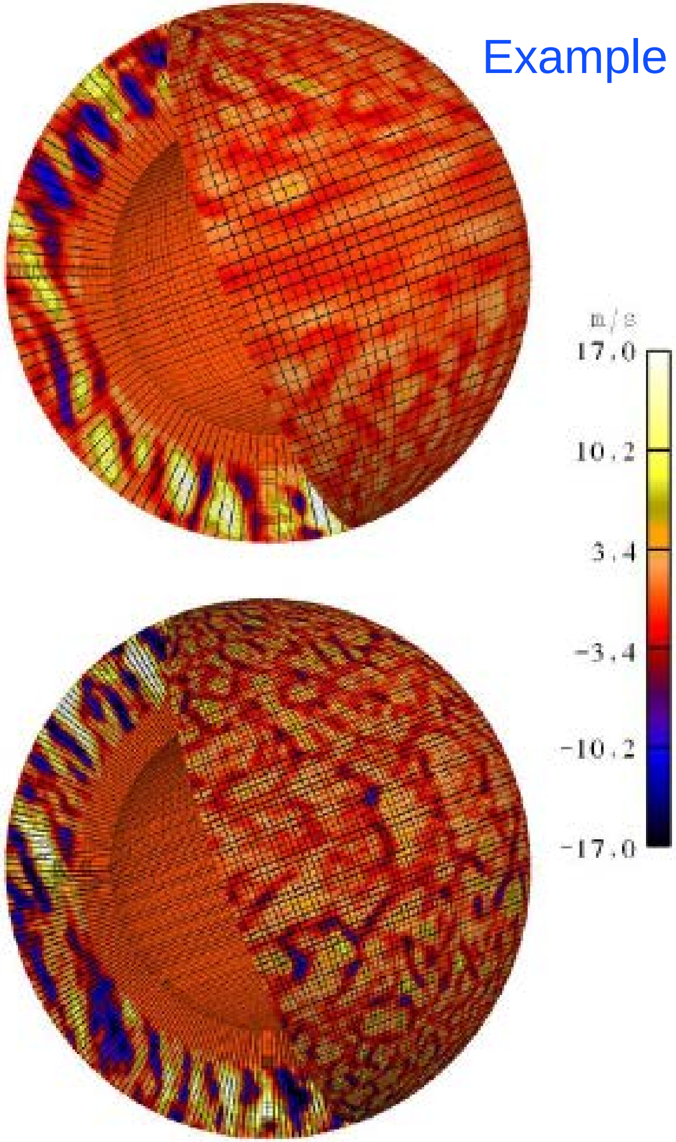
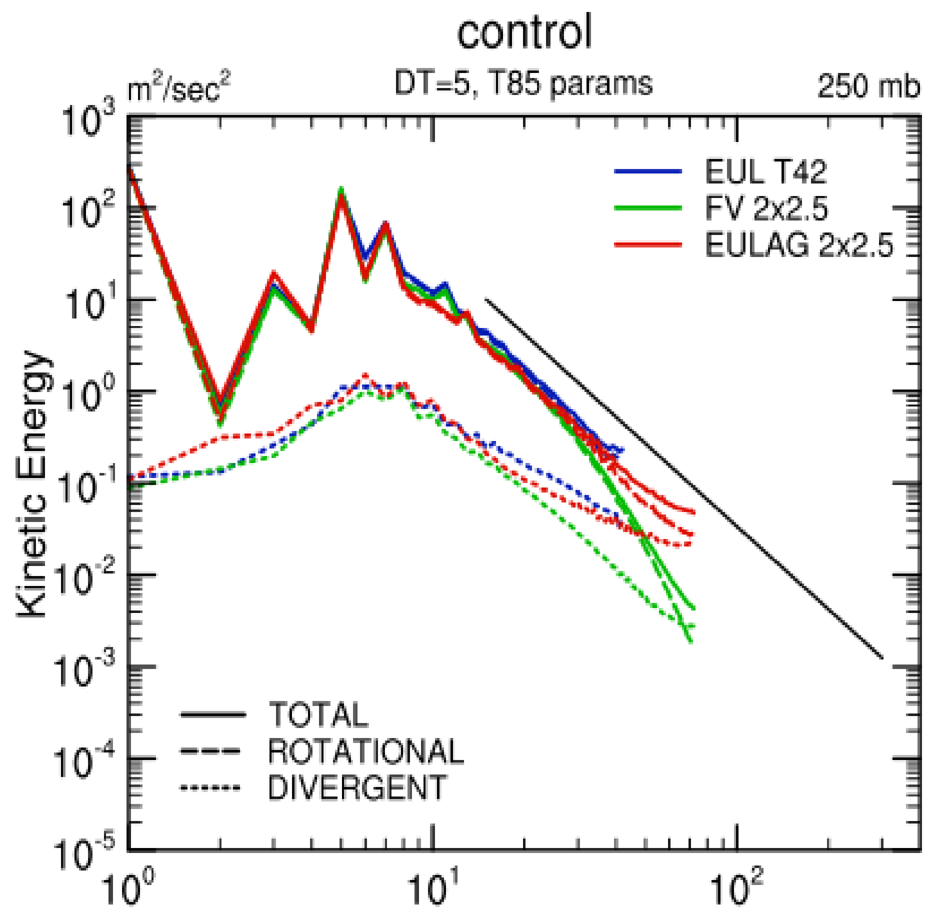


Fig. 1. Color rendering of the radial component of the flow velocity in two simulations carried out on a longitude-latitude-radius spatial mesh of size $128 \times 64 \times 47$ (top) and $256 \times 128 \times 96$ (bottom), with the mesh lines overlaid as black lines. Both snapshots are taken many tens of convective eddy turnover times after convection has reached a statistically stationary state. The tendencies for convective flow structures to align themselves parallel to the rotation axis at low latitudes is typical of global simulations of solar convection, with or without magnetic fields, operating in this parameter regime (see, e.g., Fig. 1 in [6]; Fig. 2 in [28]).



ECMWF