Compatible finite element methods for numerical weather prediction

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Talk Outline

- Definition of 1D compatible finite element spaces.
- Absence of spurious pressure modes.
- Definition and examples of 2D compatible finite element spaces.
- Absence of spurious pressure modes, existence of steady geostrophic pressure modes.
- Energy-enstrophy conservation, PV conservation for nonlinear shallow-water equations.
- Some numerical tests.

Why compatible finite elements?

Compatible finite elements

Also known as:

- discrete differential forms (Bossavit, electromagnetism),
- Finite element exterior calculus (Arnold, elasticity).

Extends properties of C-grid but extra flexibility allows:

1. Higher-order consistency on arbitrary meshes,
2. Flexibility to alter DOF ratio between velocity and pressure,
3. No orthogonality constraint on meshes.

Building finite element methods

Discretise the 1D wave equation:

$$u_t + p_x = 0, \quad p_t + u_x = 0, \quad u(0) = u(1),\, p(0) = p(1).$$

The starting point

Represent u and p in some finite element space.

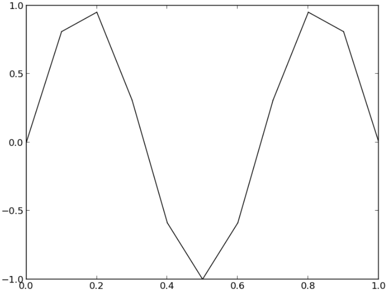
How are different finite element spaces defined?

Finite element spaces are defined by:

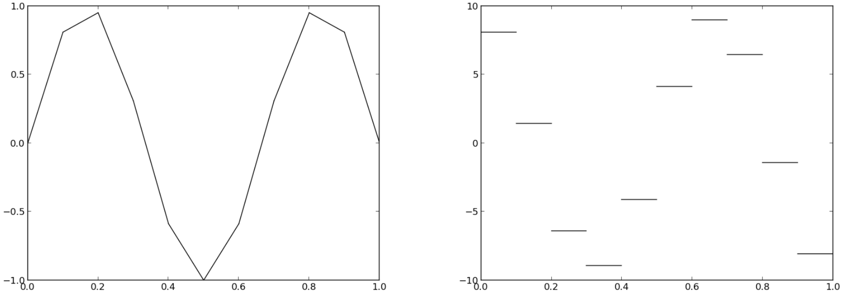
1. The type of polynomials used in each element,
2. The degree of continuity across element boundaries.

Example:

1. p and u both linear functions in each element
2. p and u both continuous across element boundaries



Incompatibility



Problem I: incompatibility

Writing $u_t + p_x = 0$ doesn't work because:

1. u is linear and continuous (P1).
2. p_x is constant and discontinuous (P0).

Projection

Approximation

Solution to incompatibility is to approximate p_x by the function v that is closest to p_x in P1 (measured using L_2 norm).

$$v = \operatorname{argmin}_{v \in P1} \|v - p_x\|_{L_2}^2 = \int_0^1 (v - p_x)^2 \, dx.$$

$$\text{Variational calculus} \implies \int_0^1 wv \, dx = \int_0^1 wp_x \, dx, \quad \forall w \in P1.$$

1. Expand w and v in a basis for P1.
2. Solve the resulting sparse matrix system for basis coefficients of v.

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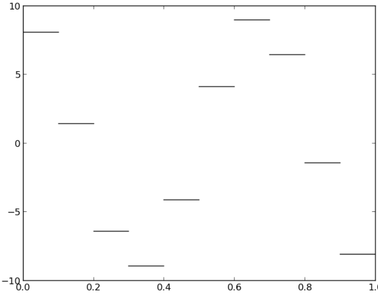

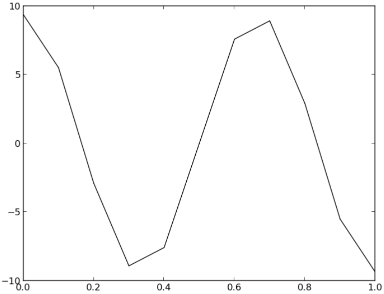

Building the equations

Unapproximated equations:

$$u_t + p_x = 0, \quad p_t + u_x = 0.$$

Minimisation:

$$u_t = \mathrm{argmin}_{u_t \in P1} \int_0^1 (u_t + p_x)^2 \, \mathrm{d}x, \quad p_t = \mathrm{argmin}_{p_t \in P1} \int_0^1 (p_t + u_x)^2 \, \mathrm{d}x.$$

Finite element discretisation:

$$\begin{aligned} \int_0^1 w u_t \, \mathrm{d}x + \int_0^1 w p_x \, \mathrm{d}x &= 0, \quad \forall w \in P1, \\ \int_0^1 v p_t \, \mathrm{d}x + \int_0^1 v u_x \, \mathrm{d}x &= 0, \quad \forall v \in P1. \end{aligned}$$

Problem II: spurious modes

Spurious pressure modes

$p \in P1$ is a spurious pressure mode if:

1. $p \neq 0$, and
2. $\int_0^1 w p_x \, \mathrm{d}x \approx 0$, for all $w \in P1$.

- If $p \in P1$, then $p_x \in P_0$.
- The projection of p_x into P_1 averages p_x over two neighbouring elements.
- On a regular grid, a zigzig pattern in p has a vanishing discrete gradient.
- Identical to spurious pressure mode on A-grid.

Smarter choice of finite element spaces

Equal finite element spaces

The spurious pressure mode problem occurs whenever we use the same finite element space for u as p.

Mixed finite element method

A mixed finite element method uses different finite element spaces for u and p.

- Already noticed that if $u \in P1$, then $u_x \in P0$.
- Choose $u \in P1$, $p \in P0$ to try to avoid averaging.
- We say that this choice of spaces is compatible with the derivative ∂/∂_x.

Weak derivatives

- Can now solve $p_t + u_x = 0$ directly since $p_t, u_x \in P0$.
- Can't solve $u_t + p_x = 0$ directly since $P0$ functions are discontinuous.

Solution

Integrate by parts in the integral form of the equations.

$$\int_0^1 w u_t \, \mathrm{d}x + \int_0^1 w p_x \, \mathrm{d}x = 0, \quad \forall w \in P1,$$

becomes

$$\int_0^1 w u_t \, \mathrm{d}x - \int_0^1 w_x p \, \mathrm{d}x + \underbrace{[wp]_0^1}_{=0} = 0, \quad \forall w \in P1.$$

No spurious pressure modes

Proposition (Ladyzhenskaya/Babuška/Brezzi (LBB) condition for P1-P0)

There exists a grid-independent constant C such that

$$\max_{w \in P1} \frac{\left| \int_0^1 w_x p \,\mathrm{d}x \right|}{\|w_x\|_{L_2}} \geq C \|p\|_{L_2}, \quad \forall p \in P0.$$

Proof.

Choose w with $w_x = p$, then
$\max_{w \in P1} \frac{\left| \int_0^1 w_x p \,\mathrm{d}x \right|}{\|w_x\|_{L_2}} \geq \frac{\int_0^1 p^2 \,\mathrm{d}x}{\left(\int_0^1 p^2 \,\mathrm{d}x \right)^{1/2}} = \left(\int_0^1 p^2 \,\mathrm{d}x \right)^{1/2} = \|p\|_{L_2}$,
$\forall p \in P0$. Hence $C = 1$. □

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Building the equations II

Unapproximated equations:

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Finite element discretisation:

$$\begin{aligned}
\int_0^1 w u_t \, \mathrm{d}x - \int_0^1 w_x p \, \mathrm{d}x &= 0, \quad \forall w \in P1, \\
\int_0^1 v p_t \, \mathrm{d}x + \int_0^1 v u_x \, \mathrm{d}x &= 0, \quad \forall v \in P0,
\end{aligned}$$

or equivalently:

$$u_t + \tilde{\partial}_x p = 0, \quad p_t + u_x = 0.$$

Energy conservation

Proposition (Energy conservation)

The P1-P0 discretisation conserves the energy

$$E = \int_0^1 \frac{1}{2}u^2 + \frac{1}{2}p^2 \, \mathrm{d}x.$$

Proof.

$$\begin{aligned}\dot{E} &= \int_0^1 u u_t \, \mathrm{d}x + \int_0^1 p p_t \, \mathrm{d}x, \\ &= \int_0^1 u_x p \, \mathrm{d}x + \int_0^1 -p u_x \, \mathrm{d}x = 0.\end{aligned}$$

$\square$

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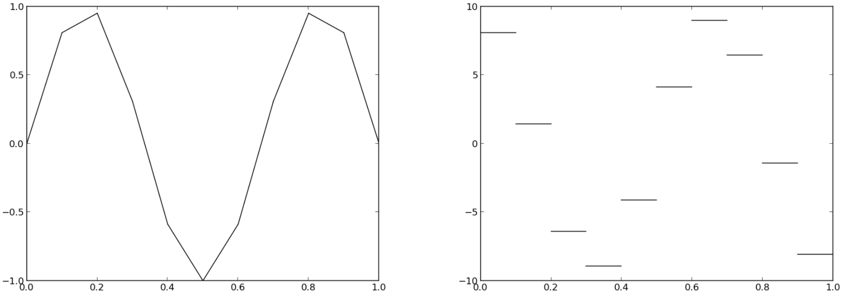
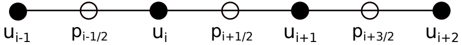
□

Connection to C-grid



Nodal basis for *P*1 and *P*0:



Equations for basis coefficients

On equispaced grid:

$$\frac{1}{6}\left(\frac{\partial u_{i-1}}{\partial t}+4\frac{\partial u_i}{\partial t}+\frac{\partial u_{i+1}}{\partial t}\right)+\frac{p_{i+1/2}-p_{i-1/2}}{\Delta x}=0,$$
$$\frac{\partial p_{i+1/2}}{\partial t}+\frac{u_{i+1}-u_i}{\Delta x}=0.$$

- Slight alteration of staggered finite difference method.
- Need to solve system of equations to get $\frac{\partial u_i}{\partial t}$.
- This modification maintains accuracy on non-equispaced grids.

General compatible finite elements in 1D

$$u \in \mathbb{V}_0, p \in \mathbb{V}_1.$$

$$\underbrace{\mathbb{V}_0}_{\text{Continuous}} \xrightarrow{\partial_x} \underbrace{\mathbb{V}_1}_{\text{Discontinuous}}$$

$$\xleftarrow{\tilde{\partial}_x}$$

General case: $\mathbb{V}_0 = Pn$, $\mathbb{V}_1 = P(n-1)_{DG}$.

Compatible finite element spaces in 2D

$$\underbrace{\mathbb{V}_0}_{\text{Continuous}} \xrightarrow{\nabla^\perp=(-\partial_y,\partial_x)} \underbrace{\mathbb{V}_1}_{\text{Continuous normals}} \xrightarrow{\nabla\cdot} \underbrace{\mathbb{V}_2}_{\text{Discontinuous}}$$

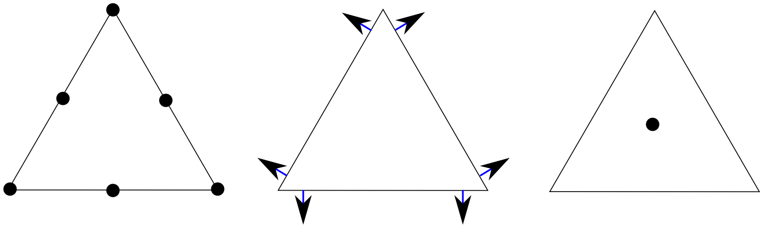
Rules:

1. $\nabla\cdot$ maps from $\mathbb{V}_1$ onto $\mathbb{V}_2$.
2. $\nabla^\perp$ maps from $\mathbb{V}_0$ onto $\ker(\nabla\cdot)$ in $\mathbb{V}_1$.

See: Arnold, Falk and Winther, Acta Numerica (2006) for history and general framework.

Example FE spaces

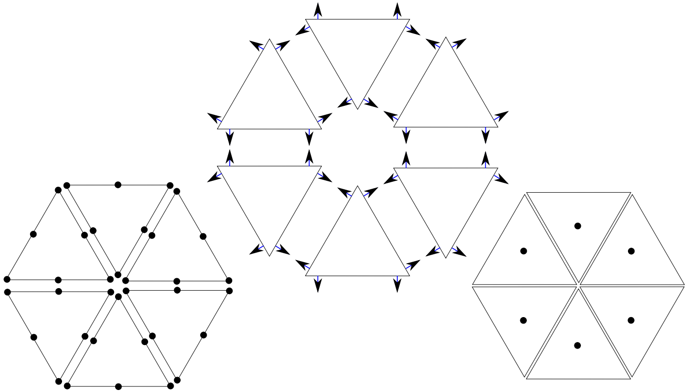
$$\underbrace{\mathbb{V}_0 = P2}_{\text{Quadratic, Continuous}} \xrightarrow{\nabla^\perp} \underbrace{\mathbb{V}_1 = BDM1}_{\text{Linear, Continuous normals}} \xrightarrow{\nabla\cdot} \underbrace{\mathbb{V}_2 = P0}_{\text{Constant, Discontinuous}}$$



Vorticity Velocity Pressure

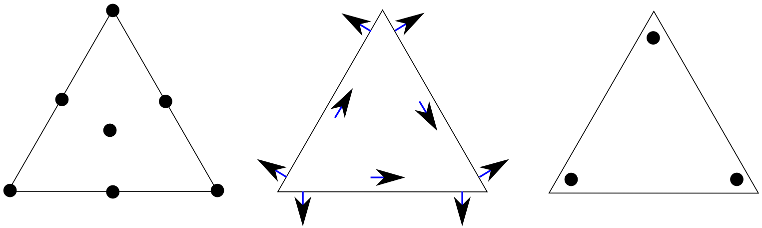
Example FE spaces

$$\underbrace{\mathbb{V}_0 = P2}_{\text{Quadratic, Continuous}} \xrightarrow{\nabla^\perp} \underbrace{\mathbb{V}_1 = BDM1}_{\text{Linear, Continuous normals}} \xrightarrow{\nabla\cdot} \underbrace{\mathbb{V}_2 = P0}_{\text{Constant, Discontinuous}}$$



Example FE spaces

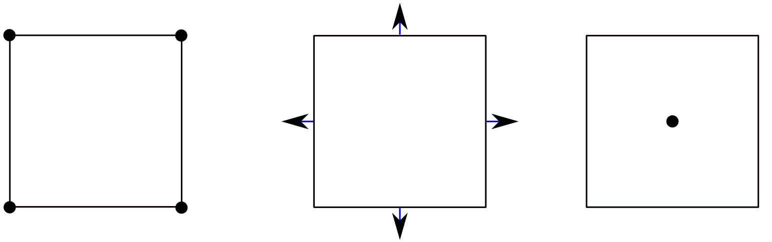
$$\underbrace{\mathbb{V}_0 = P2+}_{\text{Quadratic (+1 Cubic) Continuous}} \xrightarrow{\nabla^{\perp}} \underbrace{\mathbb{V}_1 = BDFM1}_{\text{Linear (+2 Quadratic) Cont. normals}} \xrightarrow{\nabla\cdot} \underbrace{\mathbb{V}_2 = P1_{DG}}_{\text{Linear Discontinuous}}$$



Vorticity Velocity Pressure

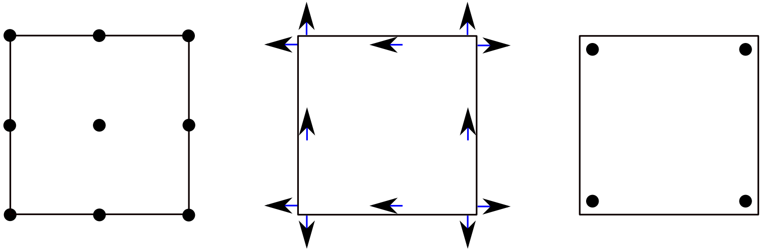
Example FE spaces

$$\underbrace{\mathbb{V}_0 = Q1}_{\text{Bilinear Continuous}} \xrightarrow{\nabla^\perp} \underbrace{\mathbb{V}_1 = RT0}_{\text{Constant/Linear, Cont. normals}} \xrightarrow{\nabla\cdot} \underbrace{\mathbb{V}_2 = Q0_{DG}}_{\text{Constant, Discontinuous}}$$

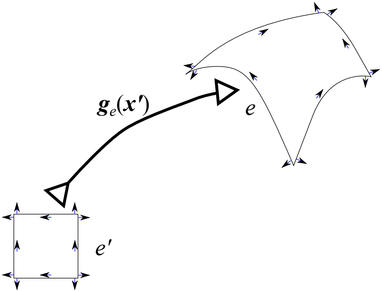


Example FE spaces

$$\underbrace{\mathbb{V}_0 = Q2}_{\text{Biquadratic Continuous}} \xrightarrow{\nabla^\perp} \underbrace{\mathbb{V}_1 = RT1}_{\text{Bilinear/Biquadratic, Cont. normals}} \xrightarrow{\nabla\cdot} \underbrace{\mathbb{V}_2 = Q1_{DG}}_{\text{Bilinear, Discontinuous}}$$



Construction of $\mathbb{V}_1$



$\boldsymbol{g}_e : e' \to e$, $\boldsymbol{x} = \boldsymbol{g}_e(\boldsymbol{x}')$.

Definition (Piola transformation)

The Piola transformation $\boldsymbol{u}' \mapsto \boldsymbol{u}$:

$$\boldsymbol{u} \circ \boldsymbol{g}_e = \frac{1}{\det \frac{\partial \boldsymbol{g}_e}{\partial \boldsymbol{x}'}} \frac{\partial \boldsymbol{g}_e}{\partial \boldsymbol{x}'} \boldsymbol{u}'$$

$$\boldsymbol{u}' \cdot \boldsymbol{n}' \, \mathrm{d}x' = \boldsymbol{g}_e^*(\boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}x)$$

Normal components

$\int_f \boldsymbol{u}' \cdot \boldsymbol{n}' \, \mathrm{d}x = \int_{\boldsymbol{g}_e(f)} \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}x.$

M. Rognes, CJC, D. Ham and A. McRae, *Automating the solution of PDEs on the sphere and other manifolds* (GMDD).

Dual operators and projections

$$\mathbb{V}_0 \quad \underset{\tilde{\nabla}^{\perp}\cdot=(-\partial_y,\partial_x)\cdot}{\overset{\nabla^{\perp}=(-\partial_y,\partial_x)}{\rightleftarrows}} \quad \mathbb{V}_1 \quad \underset{\tilde{\nabla}}{\overset{\nabla\cdot}{\rightleftarrows}} \quad \mathbb{V}_2$$

where (assume no boundaries)

$$\begin{aligned}
\int_{\Omega} \boldsymbol{w}\cdot\tilde{\nabla} h \,\mathrm{d}x &= -\int_{\Omega} \nabla\cdot\boldsymbol{w} h \,\mathrm{d}x, \quad \forall \boldsymbol{w}\in\mathbb{V}_1 \\
\int_{\Omega} \gamma\tilde{\nabla}^{\perp}\cdot\boldsymbol{u} \,\mathrm{d}x &= -\int_{\Omega} \nabla^{\perp}\gamma\cdot\boldsymbol{u} \,\mathrm{d}x, \quad \forall \gamma\in\mathbb{V}_0.
\end{aligned}$$

Dual operators and projections

$$
\begin{array}{ccccc}
\mathbb{V}_0 & \xrightarrow{\nabla^\perp=(-\partial_y,\partial_x)} & \mathbb{V}_1 & \xrightarrow{\nabla\cdot} & \mathbb{V}_2 \\
& \xleftarrow{\tilde{\nabla}^\perp\cdot=(-\partial_y,\partial_x)\cdot} & & \xleftarrow{\tilde{\nabla}} &
\end{array}
$$

Also define projections Π_i into $\mathbb{V}_i$, $i = 0, 1, 2$ by:

$$
\begin{aligned}
\int_\Omega \gamma(\Pi_0\alpha)\,\mathrm{d}x &= \int \gamma\alpha\,\mathrm{d}x, \quad \forall\gamma \in \mathbb{V}_0, \\
\int_\Omega \boldsymbol{w}\cdot(\Pi_1\boldsymbol{F})\,\mathrm{d}x &= \int \boldsymbol{w}\cdot\boldsymbol{F}\,\mathrm{d}x, \quad \forall\boldsymbol{w} \in \mathbb{V}_1, \\
\int_\Omega \phi(\Pi_2\psi)\,\mathrm{d}x &= \int \phi\psi\,\mathrm{d}x, \quad \forall\phi \in \mathbb{V}_2.
\end{aligned}
$$

Dual operators and projections

$$\mathbb{V}_0 \xrightarrow{\nabla^\perp} \mathbb{V}_1 \xrightarrow{\nabla\cdot} \mathbb{V}_2$$
$$\xleftarrow{\tilde{\nabla}^\perp\cdot} \qquad \xleftarrow{\tilde{\nabla}}$$

Properties

1. $\tilde{\nabla}^\perp \cdot \Pi_1 \boldsymbol{u}^\perp = \Pi_0 \nabla \cdot \boldsymbol{u}$ for $\boldsymbol{u} \in \mathbb{V}_1$,
2. $\Pi_1 \nabla \psi = \tilde{\nabla} \Pi_2 \psi$ for $\psi \in \mathbb{V}_0$,
3. $\tilde{\nabla}^\perp \cdot \tilde{\nabla} = 0$ (of course $\nabla \cdot \nabla^\perp = 0$).

Application to linearised RWSE

$$\boldsymbol{u}_t + f\Pi_1\boldsymbol{u}^\perp + g\tilde{\nabla}h = 0, \quad h_t + H\nabla\cdot\boldsymbol{u} = 0, \qquad \boldsymbol{u}\in\mathbb{V}_1,\ h\in\mathbb{V}_2.$$

Proposition (Ladyzhenskaya/Babuška/Brezzi (LBB) condition for 2D)

There exists a grid-independent constant C such that

$$\max_{\boldsymbol{w}\in\mathbb{V}_1}\frac{\left|\int_\Omega \nabla\cdot\boldsymbol{w}p\,\mathrm{d}x\right|}{\|\nabla\cdot\boldsymbol{w}\|_{L_2}} \geq C\|p\|_{L_2}, \quad \forall p\in\mathbb{V}_2.$$

Proof.

Choose $\boldsymbol{w}$ with $\nabla\cdot\boldsymbol{w} = p$, then $\max_{\boldsymbol{w}\in\mathbb{V}_1}\frac{\left|\int_\Omega \nabla\cdot\boldsymbol{w}p\,\mathrm{d}x\right|}{\|\nabla\cdot\boldsymbol{w}\|_{L_2}} \geq \frac{\int_\Omega p^2\,\mathrm{d}x}{\left(\int_\Omega p^2\,\mathrm{d}x\right)^{1/2}} = \left(\int_\Omega p^2\,\mathrm{d}x\right)^{1/2} = \|p\|_{L_2}$, $\forall p\in\mathbb{V}_2$. Hence $C = 1$. □

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$$\boldsymbol{u}_t + f\Pi_1 \boldsymbol{u}^\perp + g\tilde{\nabla} h = 0, \quad h_t + H\nabla\cdot\boldsymbol{u} = 0, \qquad \boldsymbol{u}\in\mathbb{V}_1,\ h\in\mathbb{V}_2.$$

Geostrophic steady states:

1. If $\nabla\cdot\boldsymbol{u} = 0$, then $\boldsymbol{u} = \nabla^\perp\psi$, $\psi\in\mathbb{V}_0$.
2. Choose $h = f\Pi_2\psi/g$, then
   $f\Pi_1(\boldsymbol{u}^\perp) = -f\Pi_1\nabla\psi = -f\tilde{\nabla}\Pi_2\psi = -g\tilde{\nabla}h.$

Application to nonlinear RSWE

$$\boldsymbol{u}_t + f\underbrace{(q\boldsymbol{u}h)}_{=\boldsymbol{Q}}^{\perp} + \nabla\left(gh + |\boldsymbol{u}|^2/2\right) = 0, \quad h_t + \nabla\cdot(\underbrace{\boldsymbol{u}h}_{=\boldsymbol{F}}) = 0.$$

$$\mapsto \boldsymbol{u}_t + f\Pi_1\boldsymbol{Q}^{\perp} + g\tilde{\nabla}\left(h + \Pi_2|\boldsymbol{u}|^2/2\right) = 0, \quad h_t + \nabla\cdot\boldsymbol{F} = 0,$$

for $\boldsymbol{u}, \boldsymbol{F} \in \mathbb{V}_1$, $h \in \mathbb{V}_2$ and some $\boldsymbol{Q}$.

Strategy from Arakawa and Lamb, Sadourny

1. Apply natural curl to get vorticity equation.
2. Map h to vertices to evaluate PV.
3. Diagnose PV flux $\boldsymbol{Q}$ via $\boldsymbol{F}$ and insert into velocity equation.

Implied PV equation

$$\boldsymbol{u}_t + f\Pi_1\boldsymbol{Q}^\perp + \tilde{\nabla}\left(gh + \Pi_2|\boldsymbol{u}|^2/2\right) = 0,$$

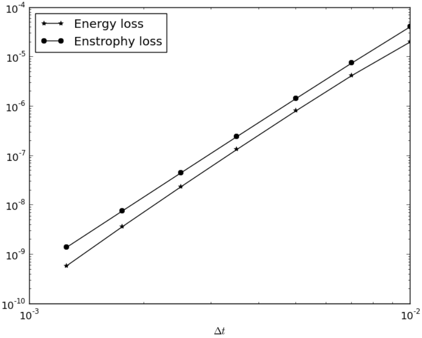
Apply $\tilde{\nabla}^\perp\cdot$: $\quad \tilde{\nabla}^\perp\boldsymbol{u}_t + \tilde{\nabla}^\perp\cdot\boldsymbol{Q}^\perp + \underbrace{\tilde{\nabla}^\perp\cdot\tilde{\nabla}}_{=0}(gh + \Pi_2|\boldsymbol{u}|^2/2) = 0.$

PV $q \in \mathbb{V}_0$ defined by $\Pi_0(qh) = \tilde{\nabla}^\perp\cdot\boldsymbol{u} + \Pi_0(f)$.

Get $\dfrac{\partial}{\partial t}\Pi_0(qh) + \tilde{\nabla}^\perp\cdot\boldsymbol{Q}^\perp = 0.$

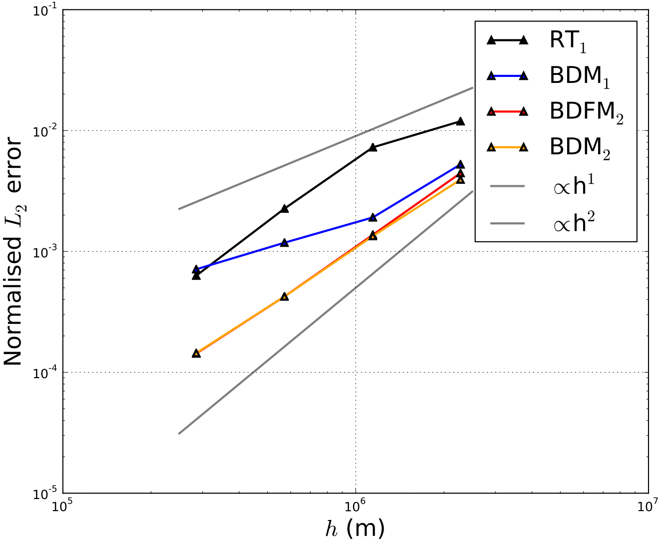
Usual continuous finite element discretisation:

$$\int_\Omega \gamma(qh)_t\,\mathrm{d}x - \int_\Omega \nabla\gamma\cdot\boldsymbol{Q}\,\mathrm{d}x = 0.$$



McRae and Cotter (submitted to QJRMS)

1. The choice $\boldsymbol{F} = \Pi_1(h\boldsymbol{u})$ and $\boldsymbol{Q} = \boldsymbol{F}q$ conserves energy and enstrophy.
2. The choice $\boldsymbol{F} = \Pi_1(h\boldsymbol{u})$ and $\boldsymbol{Q} = \boldsymbol{F}(q - (\tau/h)\boldsymbol{F} \cdot \nabla q)$ conserves energy and dissipates enstrophy (APVM).
3. Both choices preserve constant q field for any initial h.



From Andrew McRae, using energy conserving, enstrophy dissipating (APVM) formulation.

Implicit timestepping setup

$$
\begin{aligned}
\boldsymbol{u}_t + (\underbrace{\boldsymbol{u}Dq}_{\boldsymbol{Q}})^{\perp} + \nabla\left(gD + \frac{1}{2}|\boldsymbol{u}|^2\right) &= 0, \\
D_t + \nabla\cdot(\underbrace{\boldsymbol{u}D}_{\boldsymbol{F}}) &= 0.
\end{aligned}
$$

Crank-Nicholson:

$$
\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^n}{\Delta t} + \overline{\boldsymbol{Q}}^{\perp} + \nabla\left(g\overline{D} + \frac{1}{2}\overline{|\boldsymbol{u}|^2}\right) = 0.
$$

$$
\begin{aligned}
\text{Solve}\qquad & D_t + \nabla\cdot(\overline{\boldsymbol{u}}D) = 0, \\
& (qD)_t + \nabla\cdot(\overline{\overline{\boldsymbol{u}}D}q) = 0, \quad \text{from } t^n \text{ until } t^{n+1}, \\
\text{Then}\qquad & \overline{\boldsymbol{Q}} = \overline{\overline{\overline{\boldsymbol{u}}D}q}.
\end{aligned}
$$

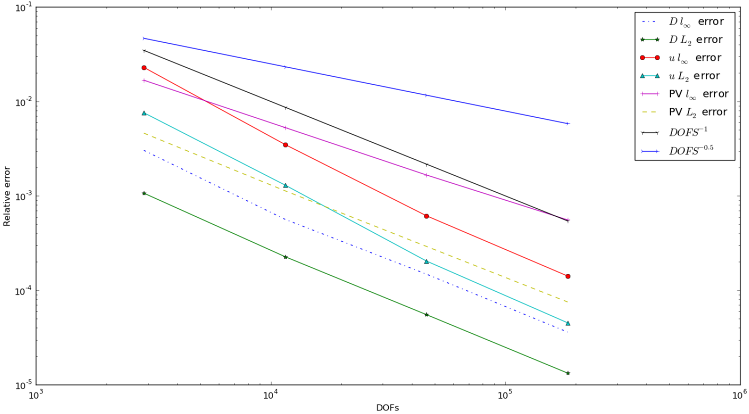
Preserves constant q fields.

Implementation details

- Scheme implemented on cubic “bendy” elements, all terms except for mass matrices are topological only (local element matrices independent of coordinates).
- P2(bubble)-BDFM1-P1DG spaces used.
- 3rd order in time SSPRK-DG used for layer depth (can locally reconstruct $\boldsymbol{F}$).
- 2 level, 3rd order in time $\bar{T}(2,3)$ Taylor-Galerkin scheme of Safjan and Oden (1995) used for PV (2 CG mass-matrix-like solves per timestep).
- 4 quasi-Newton iterations per timestep, and $\theta = 1/2$.
- Helmholtz equation formed by hybridisation.

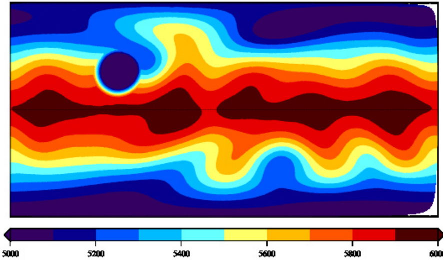
Testcases

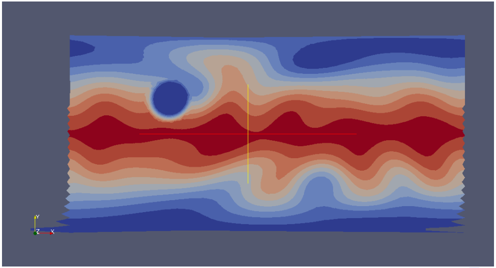
Solid rotation testcase.

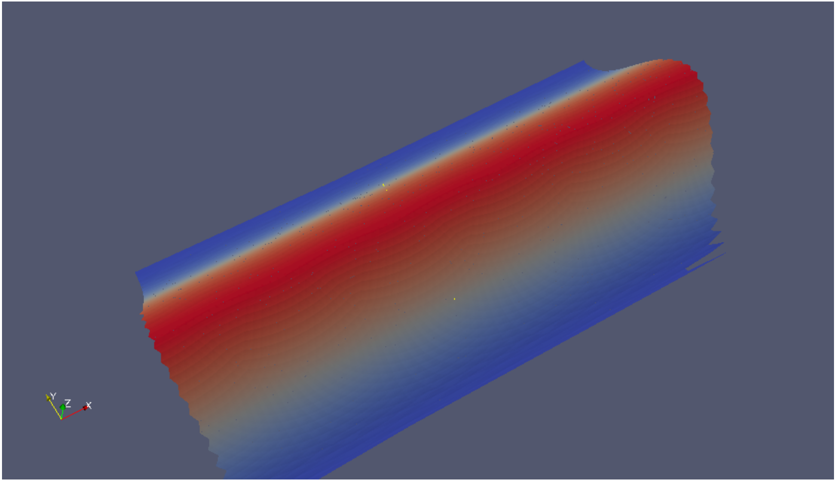


Testcases

Mountain test case (Grid 5, 46080 DOFs).







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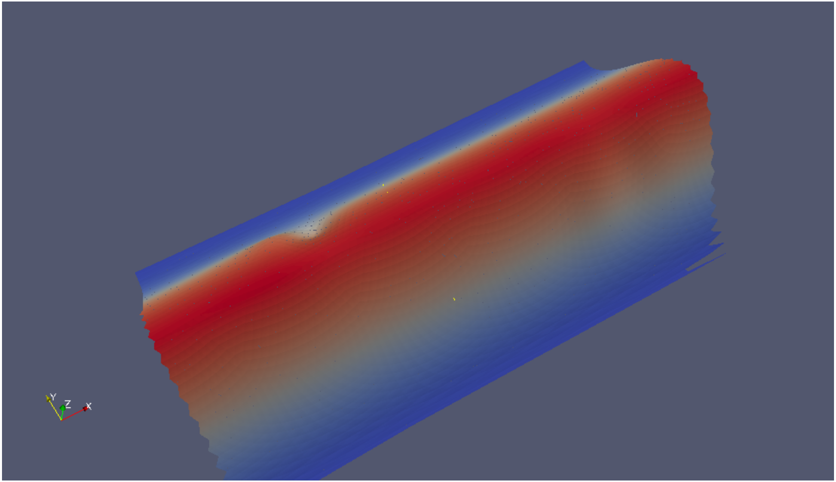

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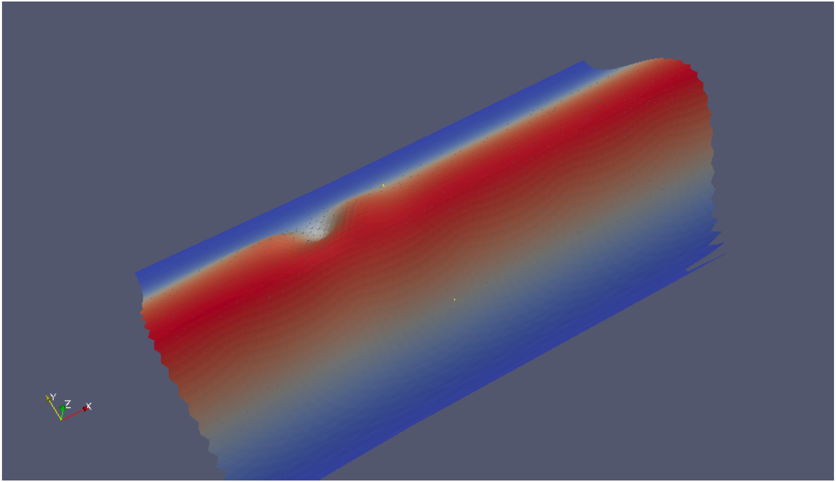

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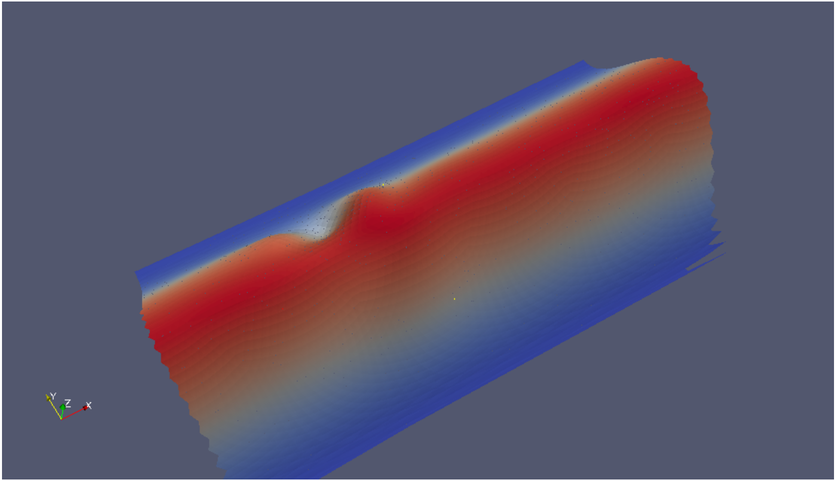

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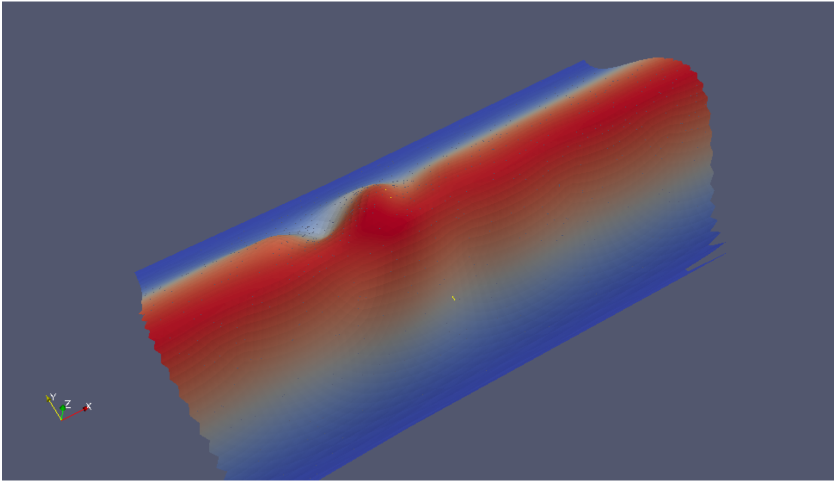

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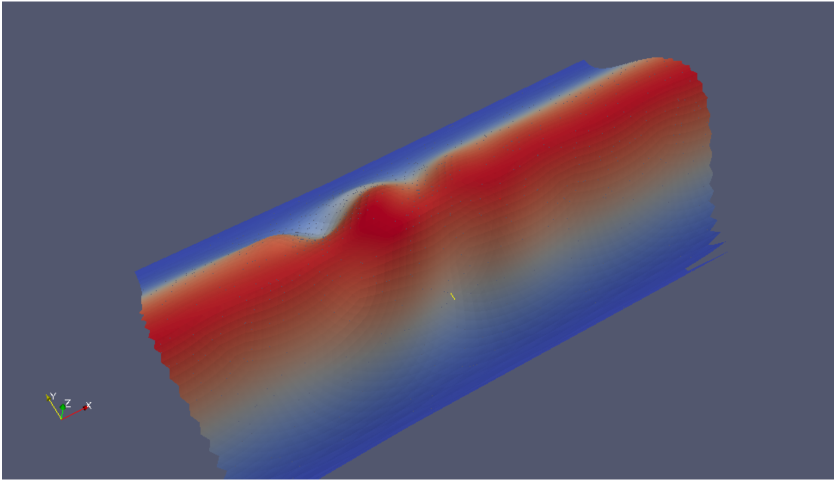

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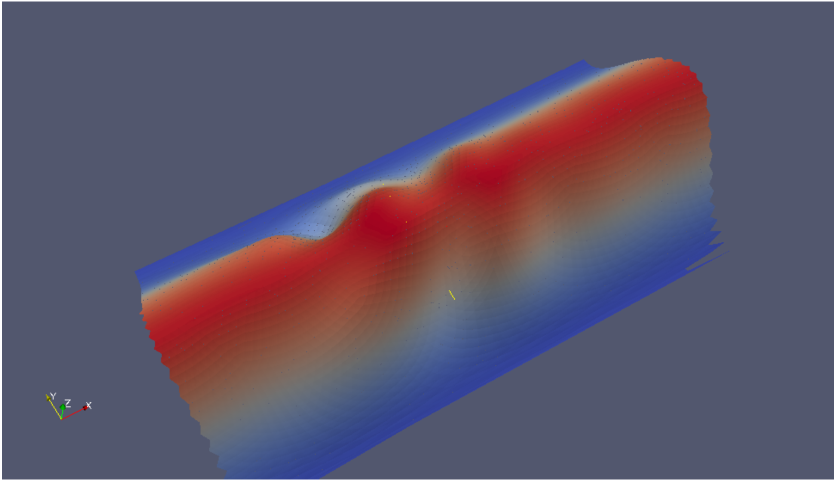

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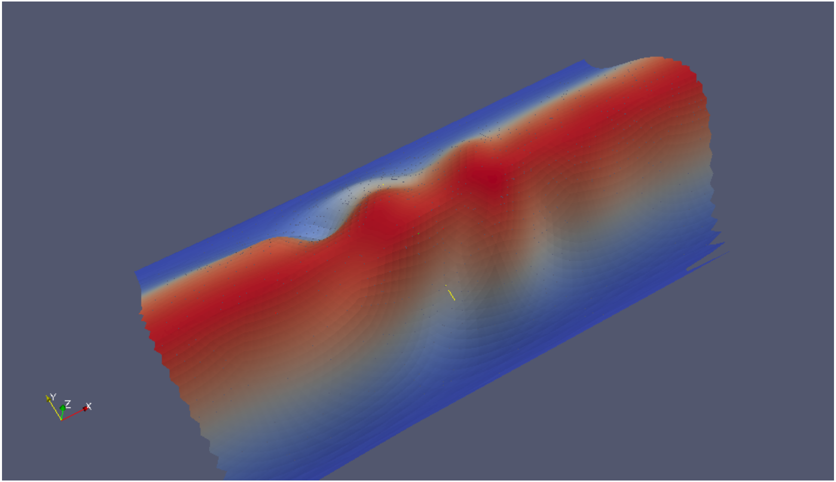

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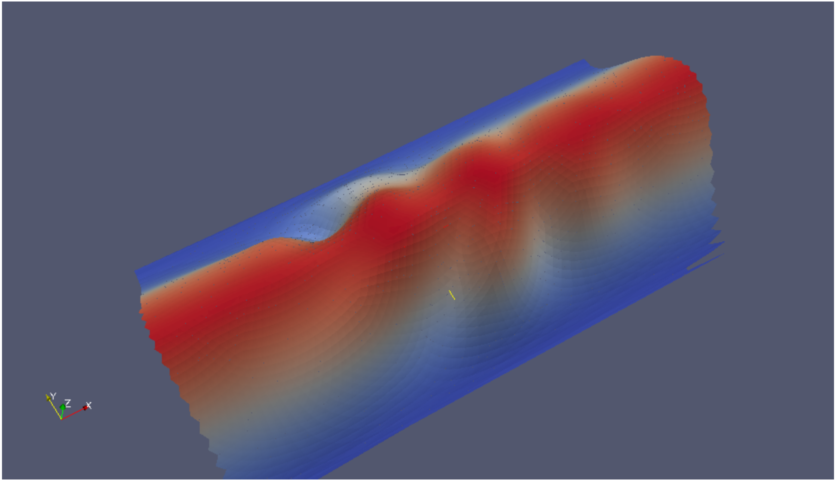

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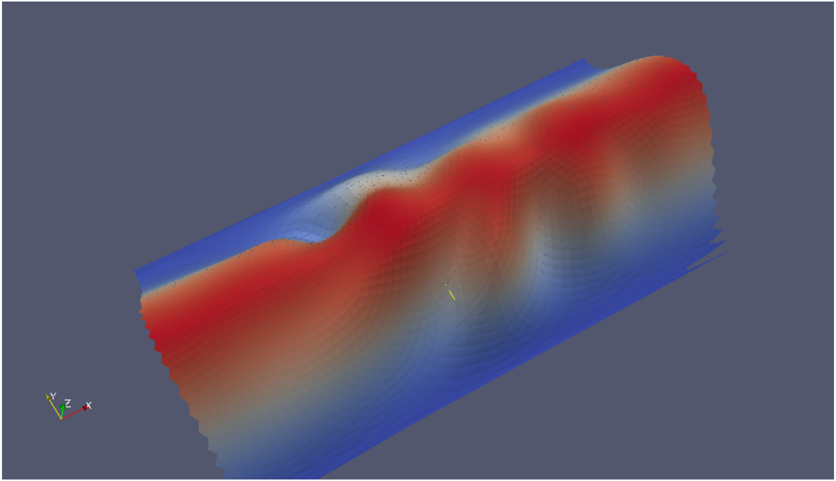

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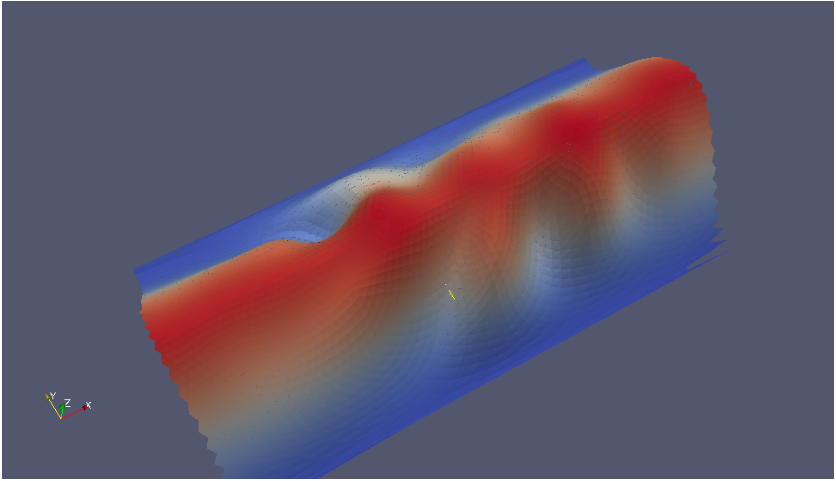

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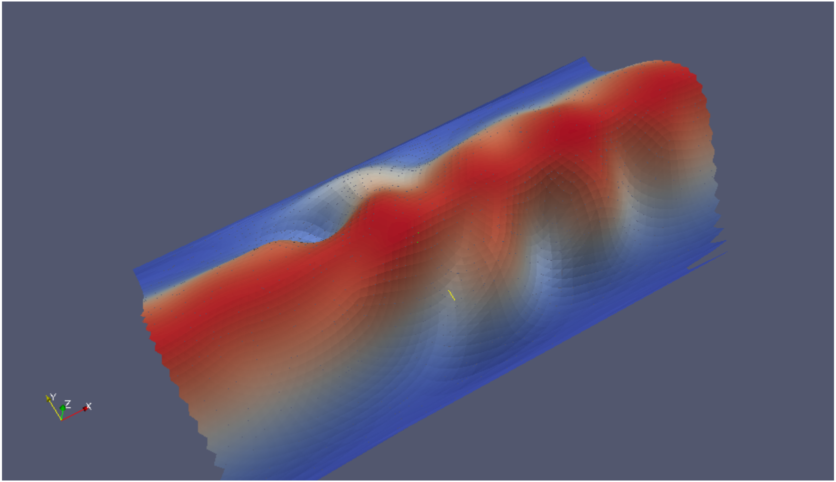

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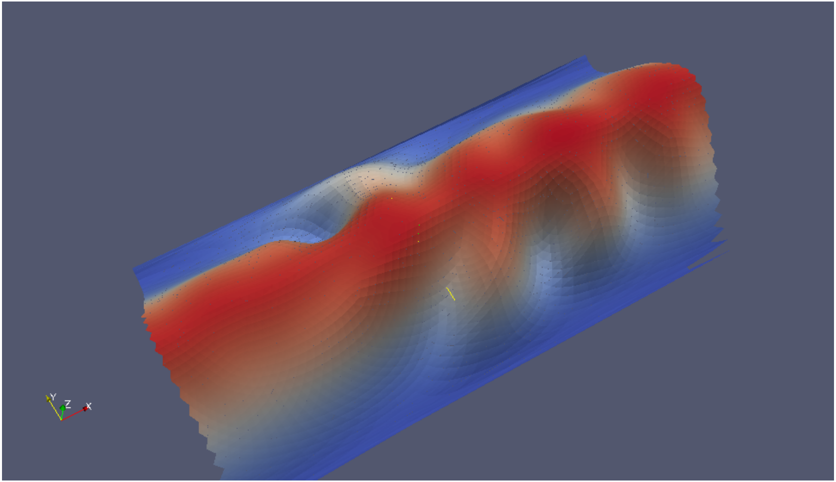

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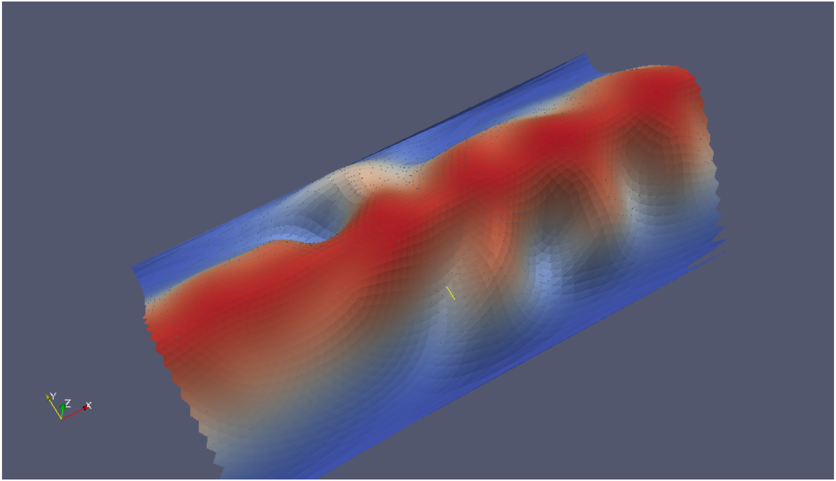

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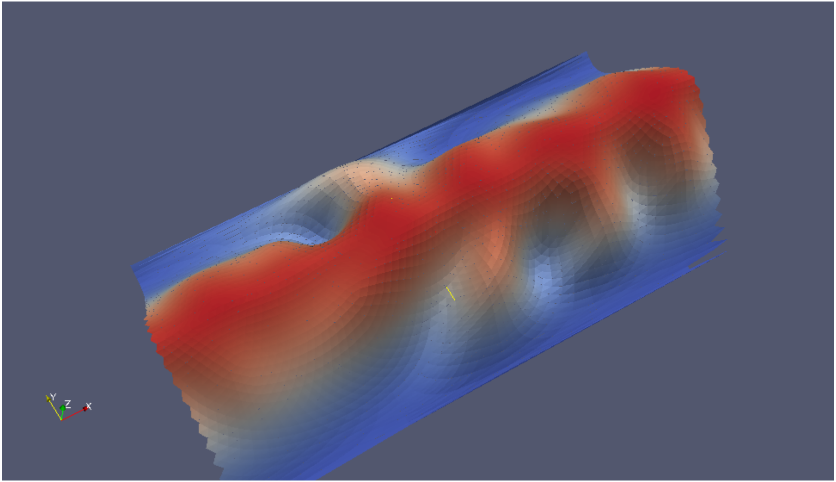

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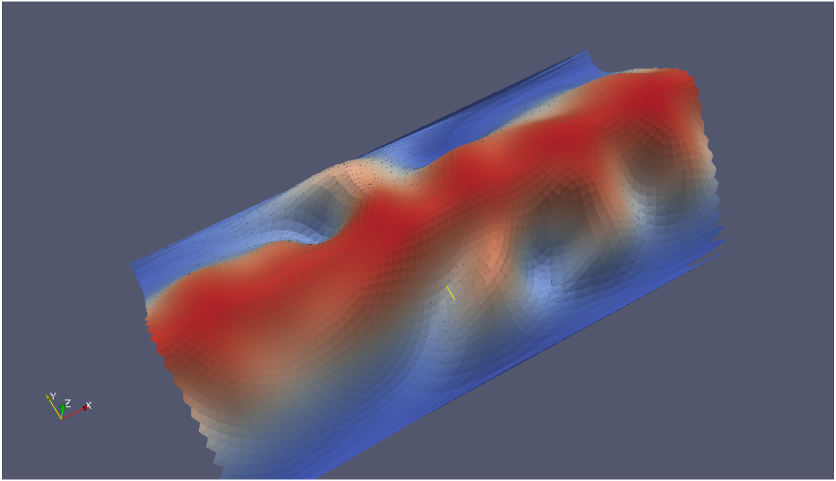

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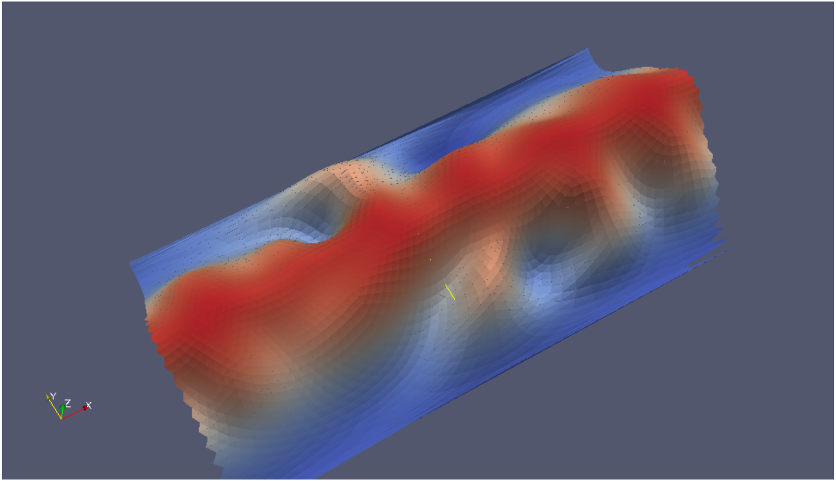

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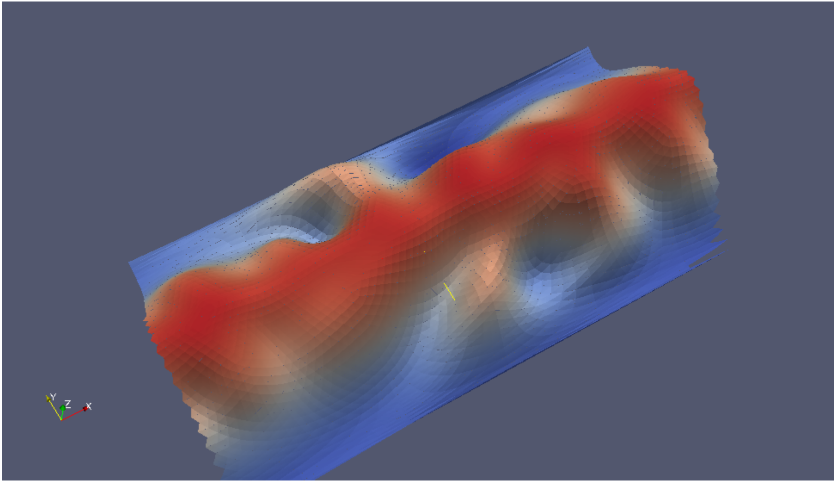

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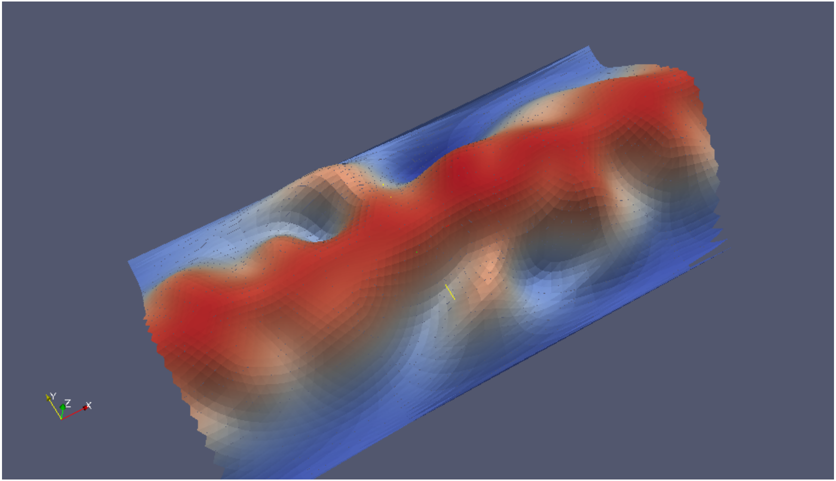

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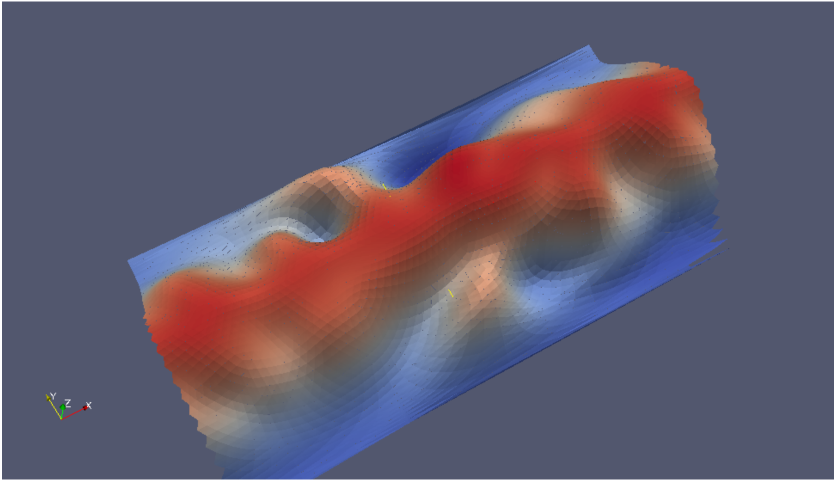
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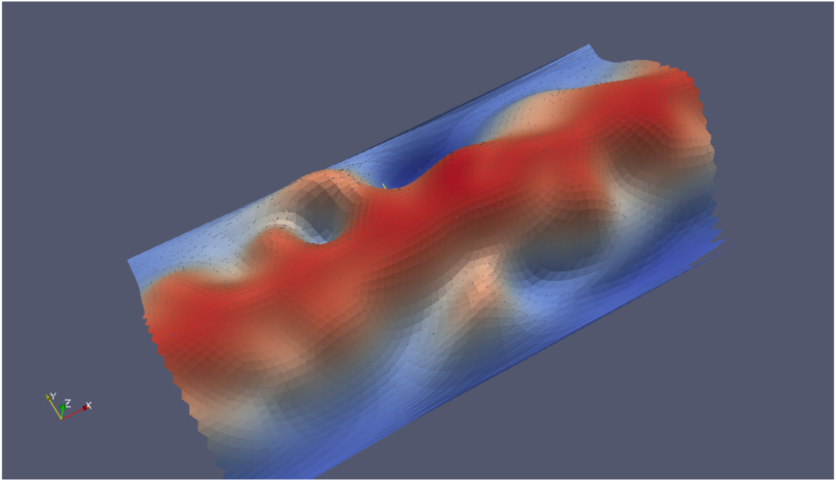

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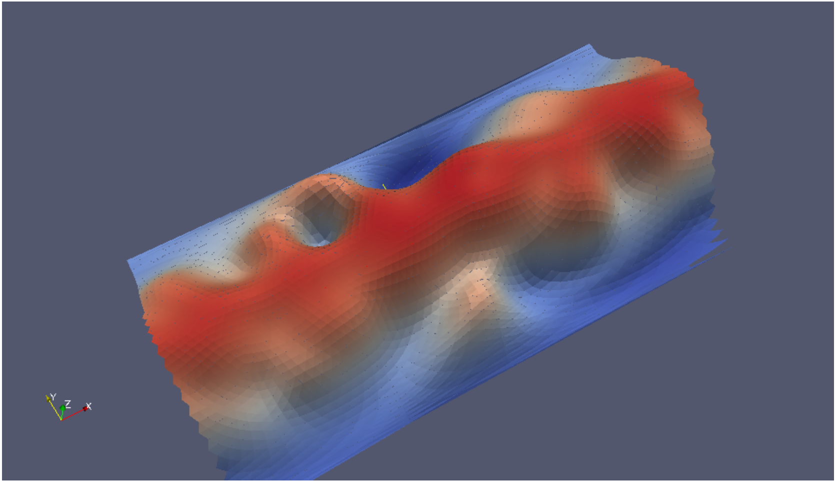

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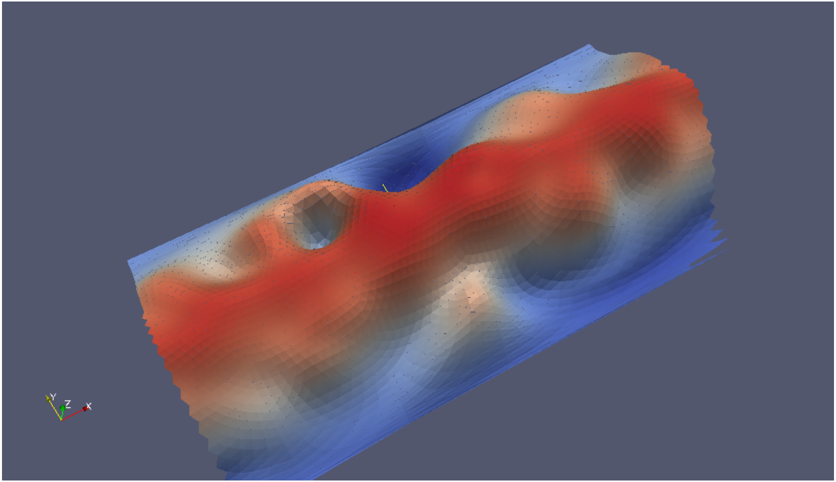

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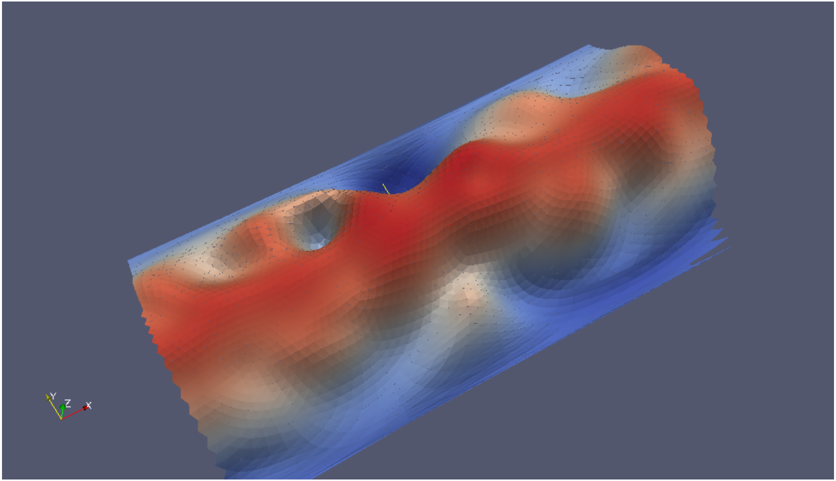

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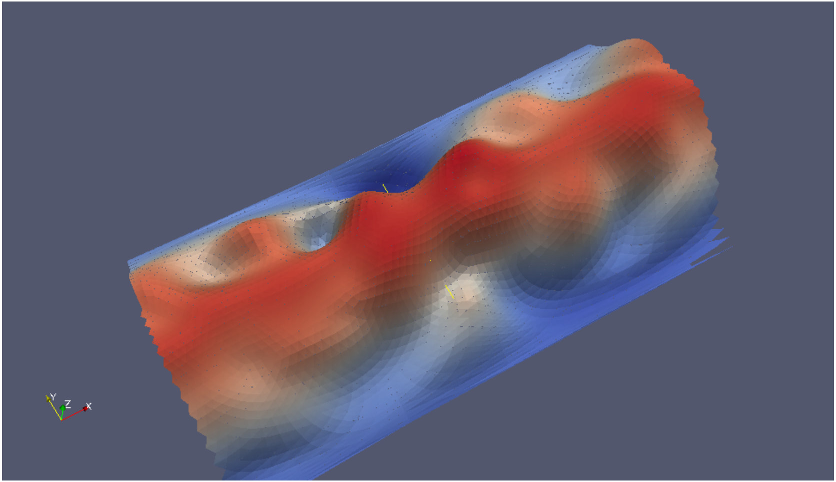

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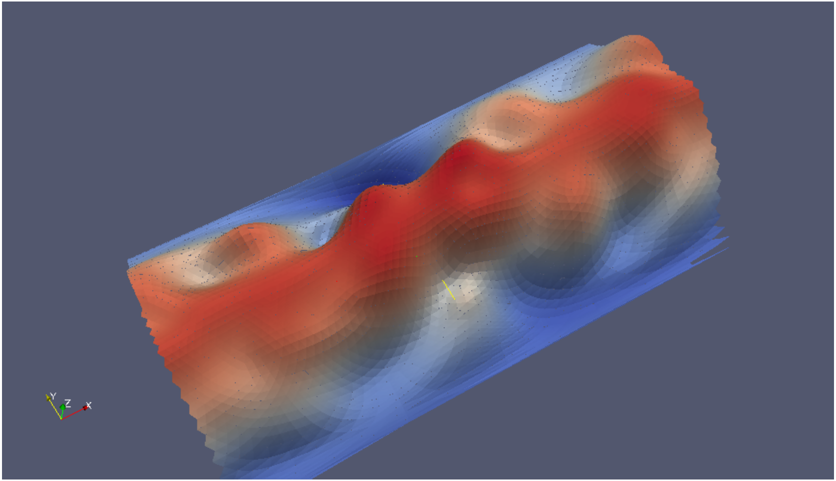

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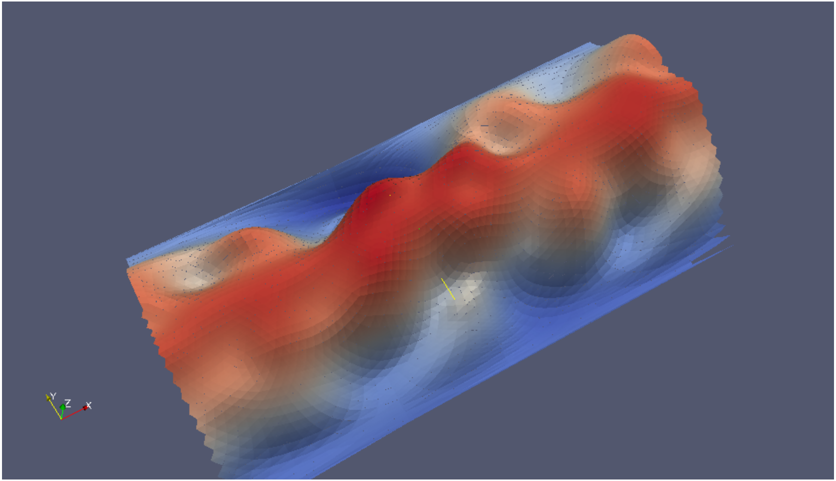
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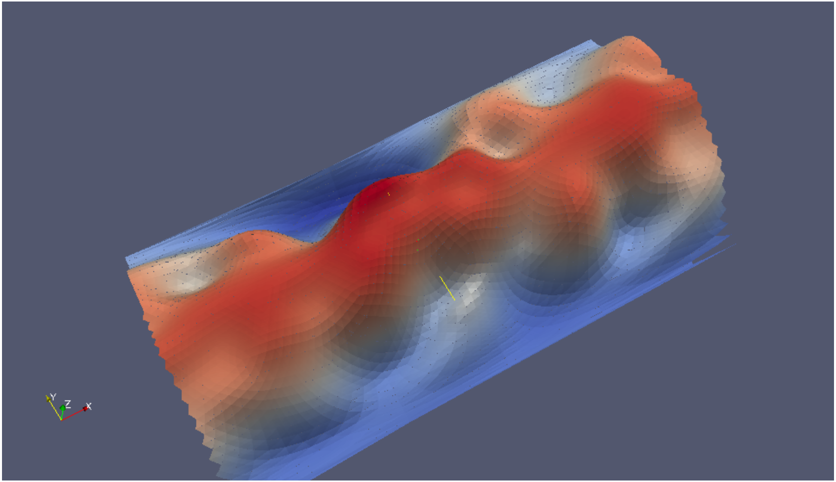

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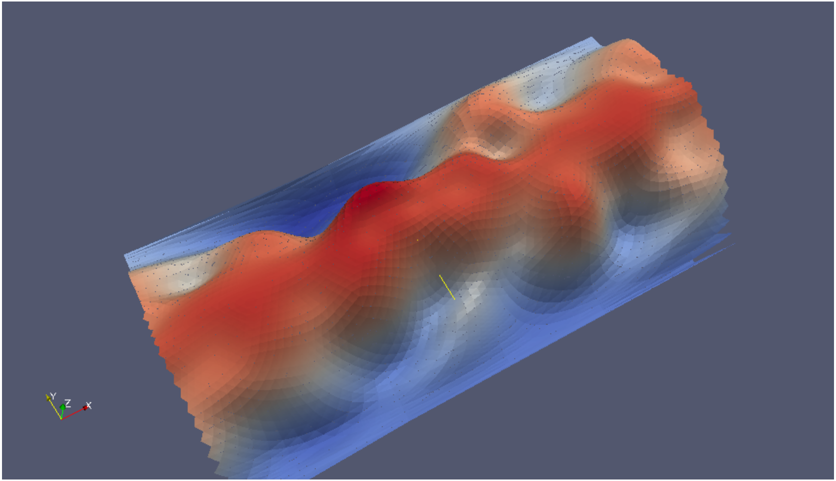

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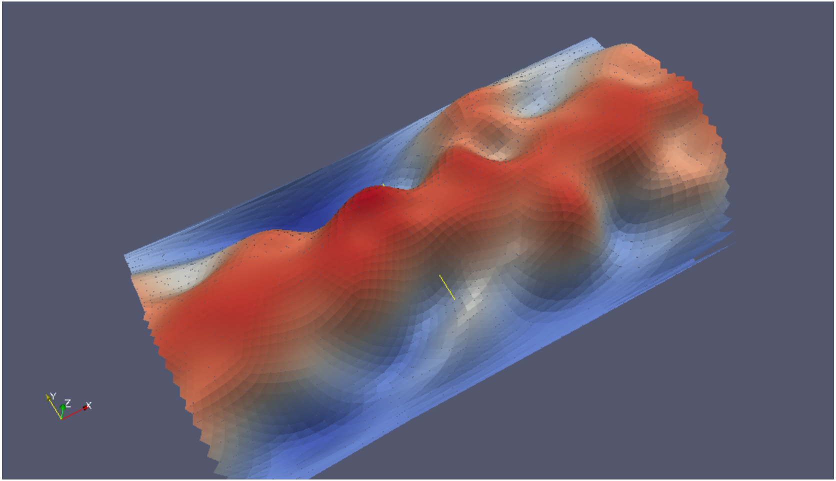

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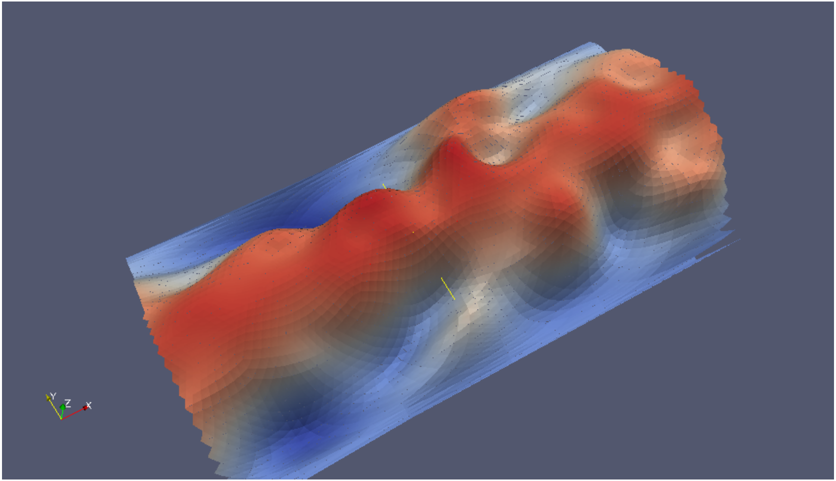

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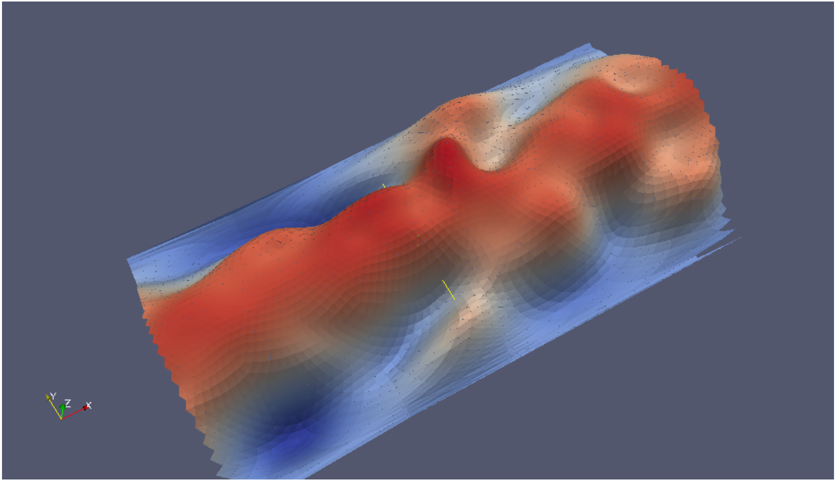

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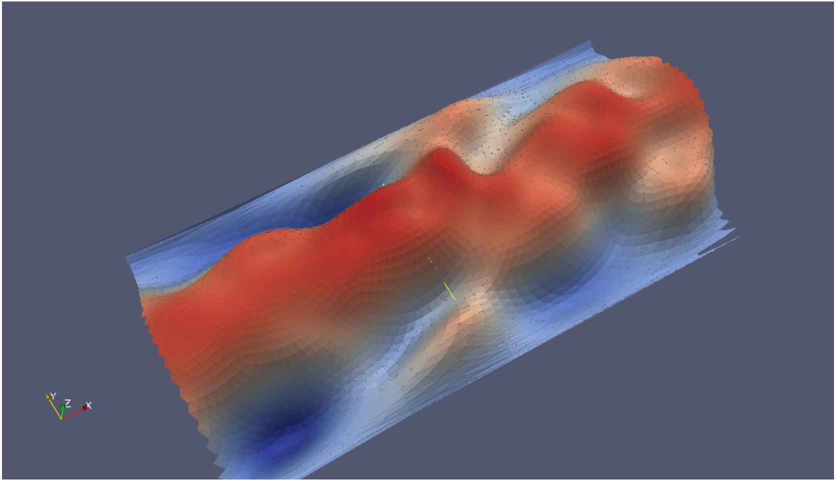

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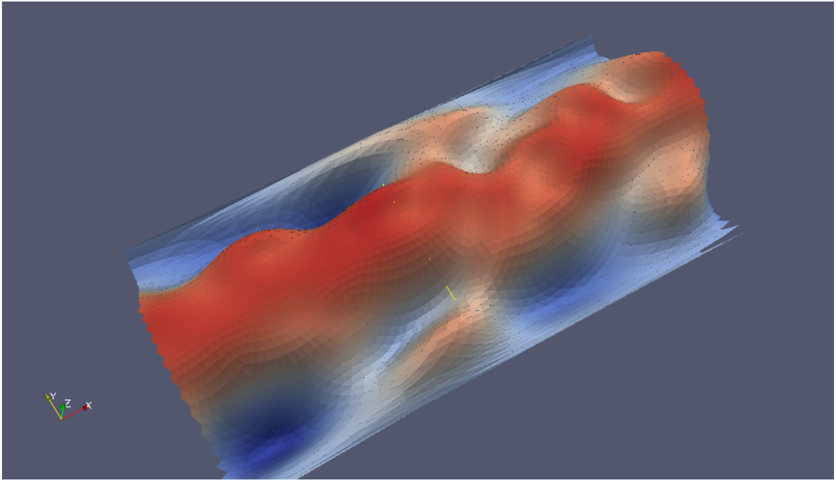

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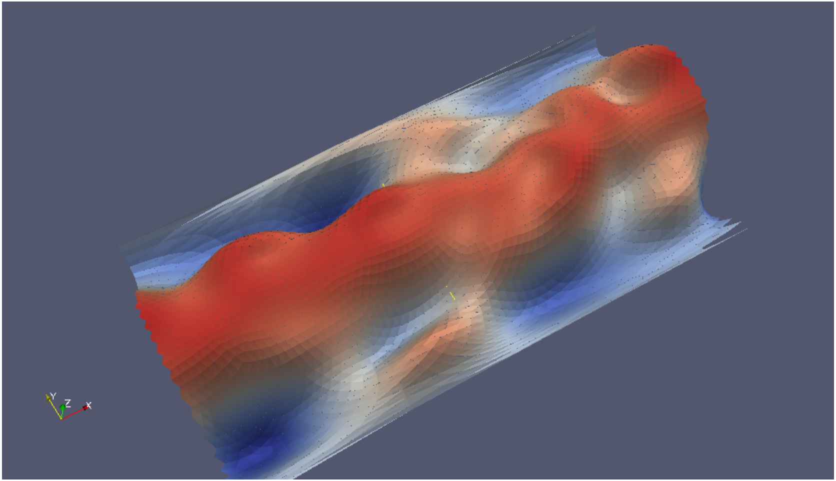

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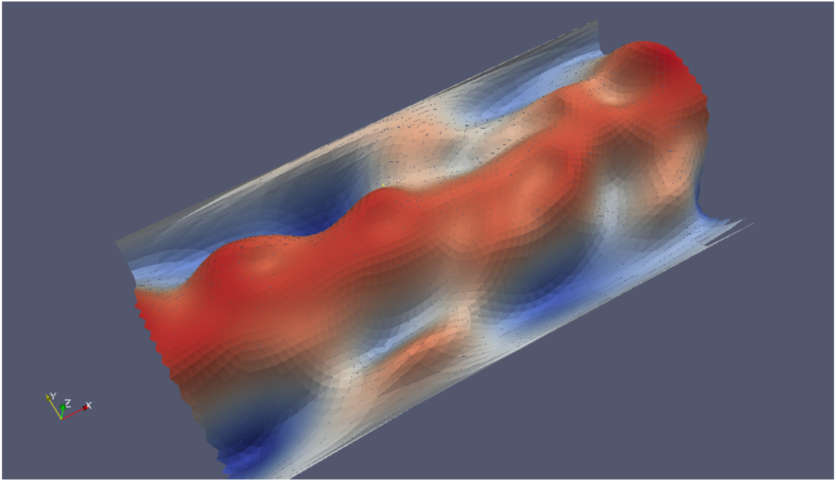

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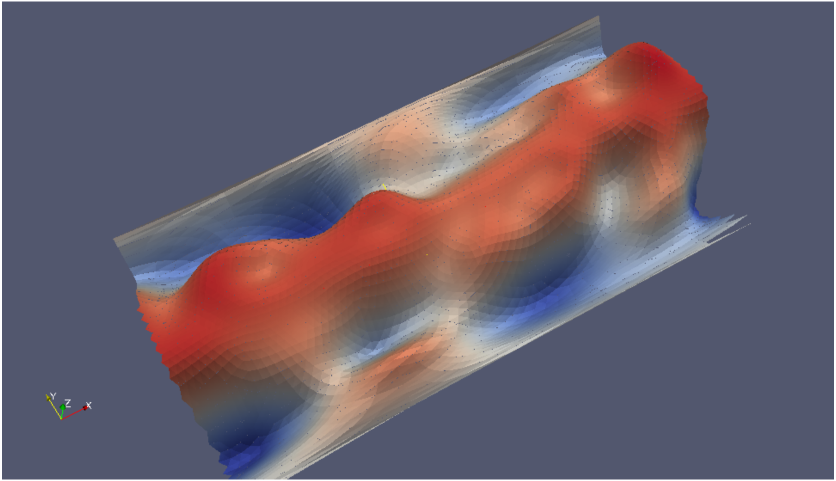

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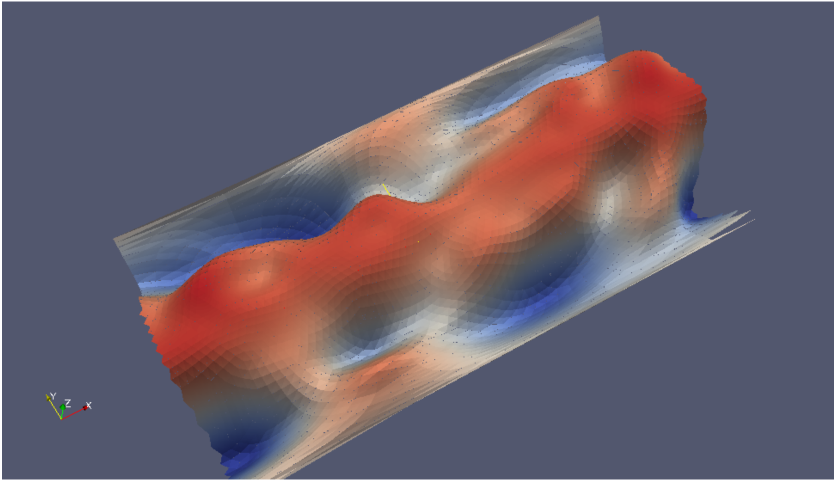

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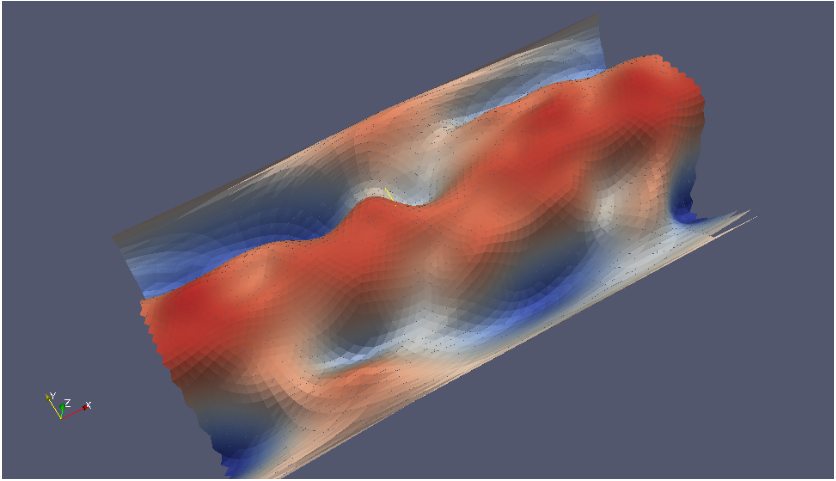

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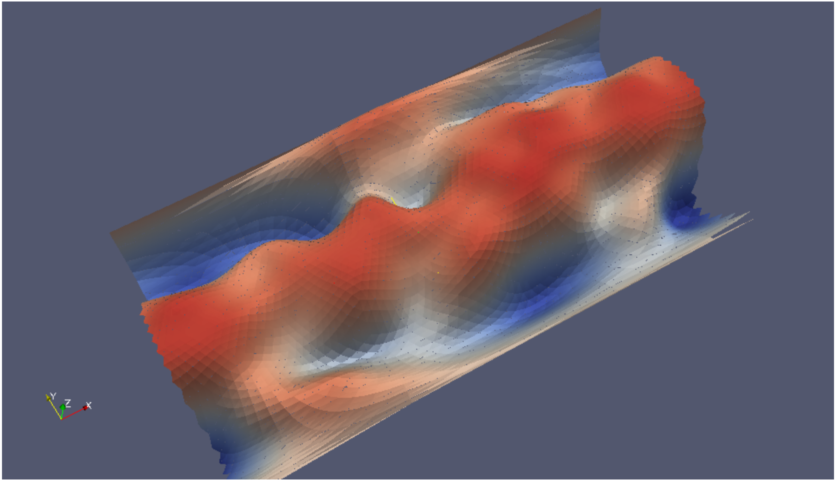
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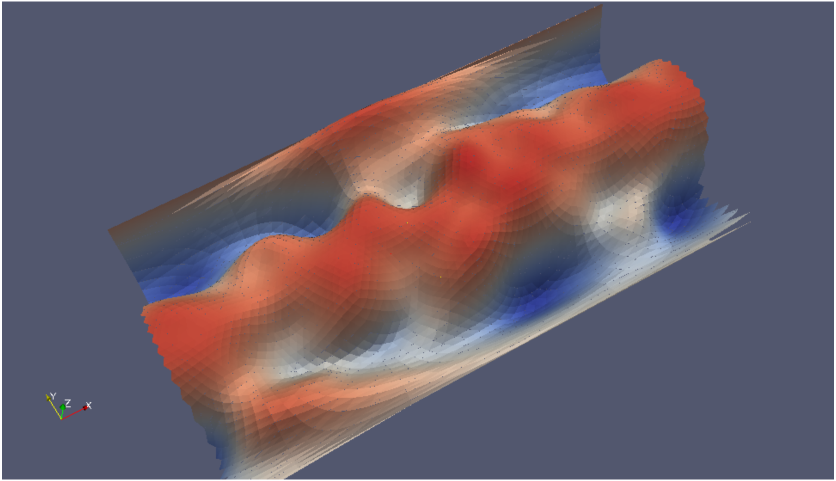
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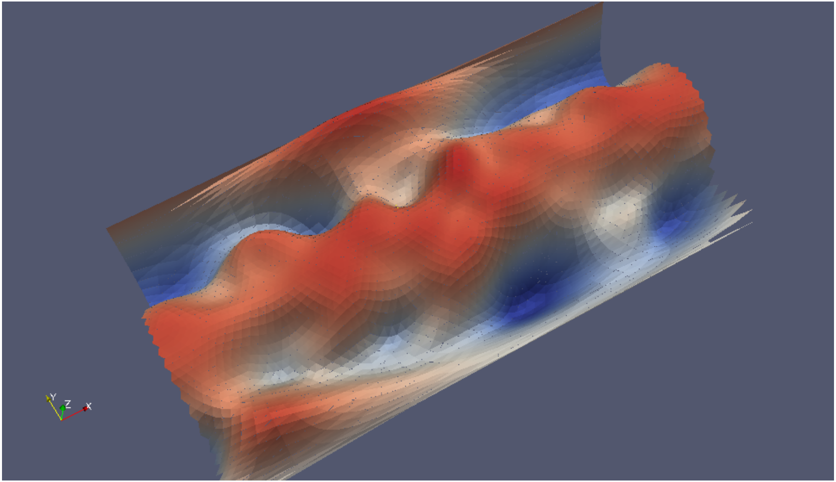

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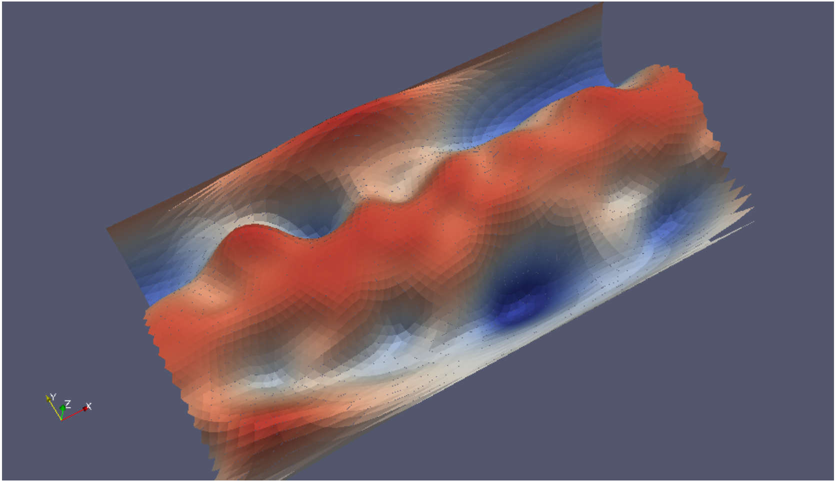

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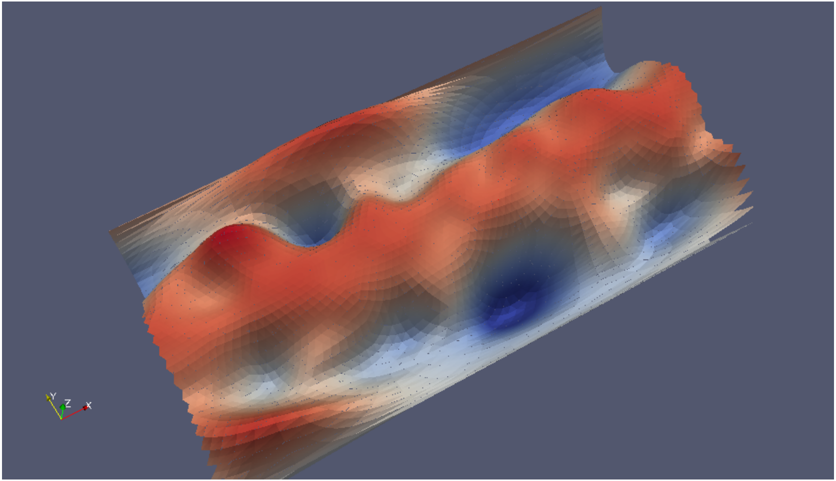
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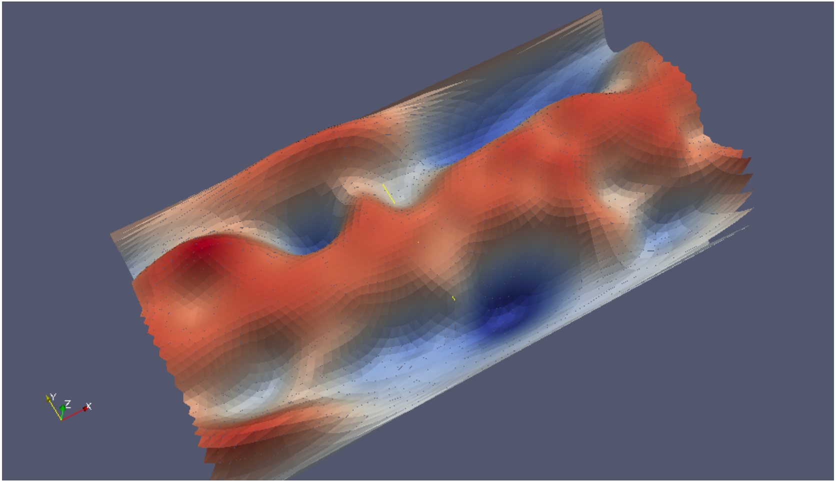

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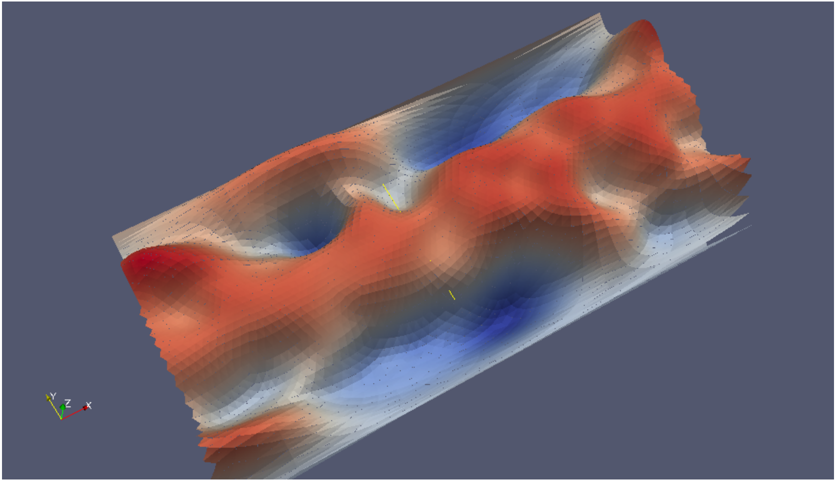

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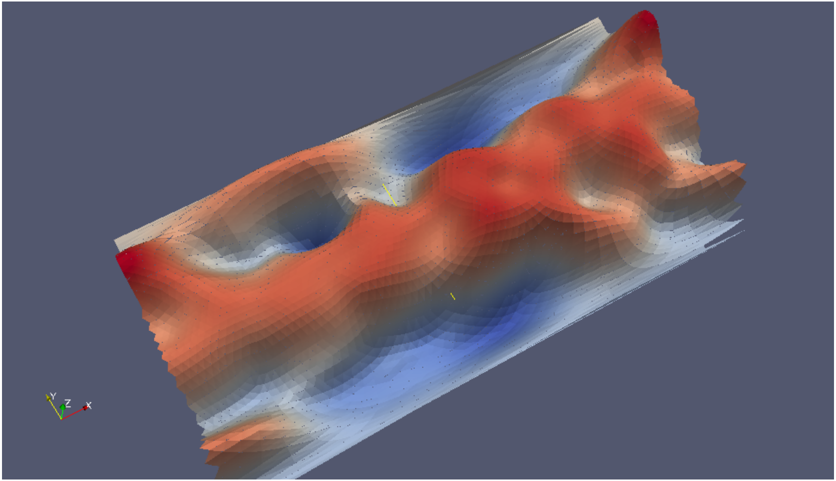
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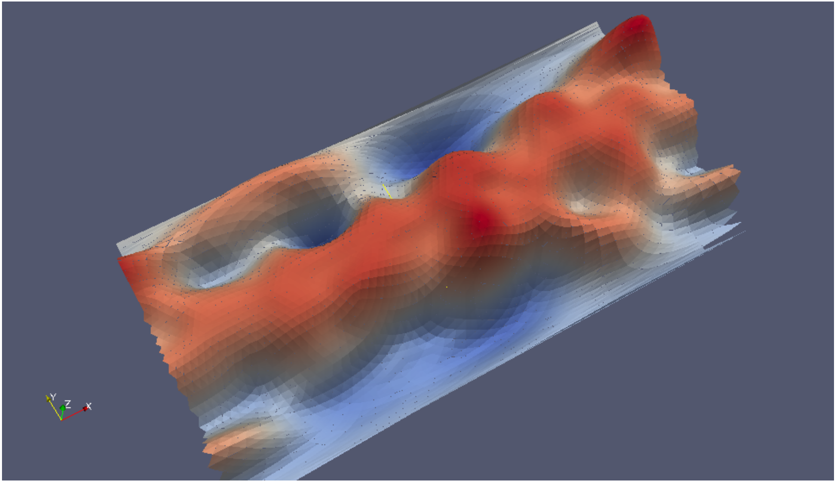
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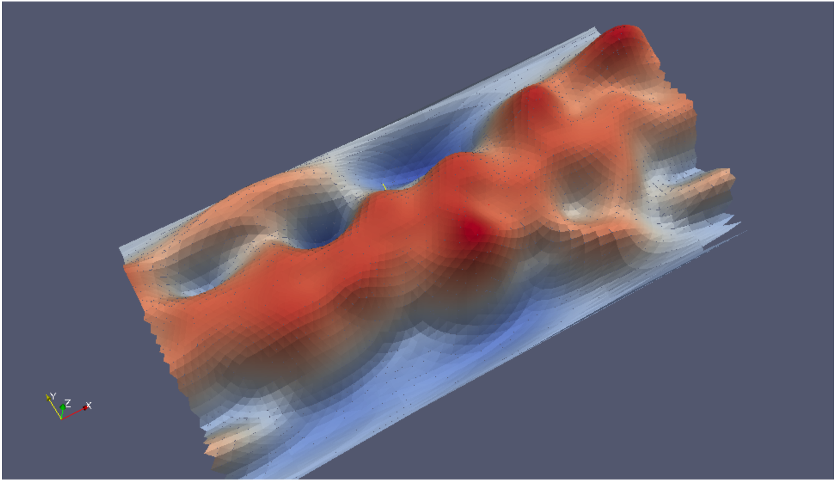
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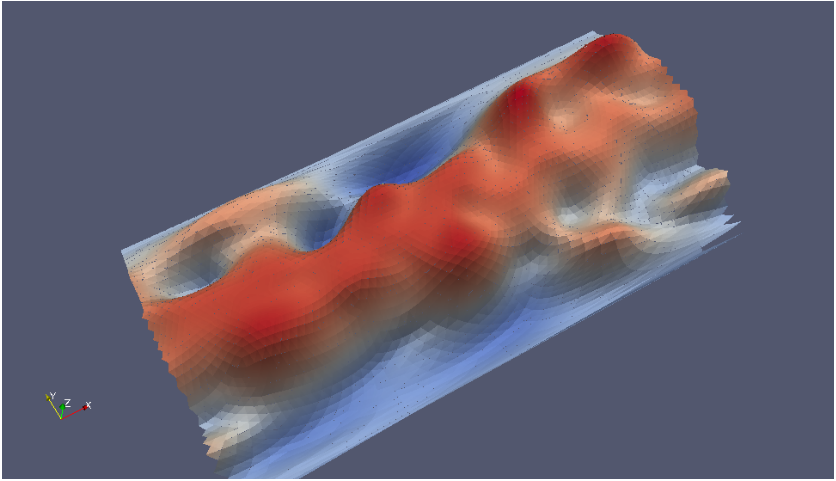

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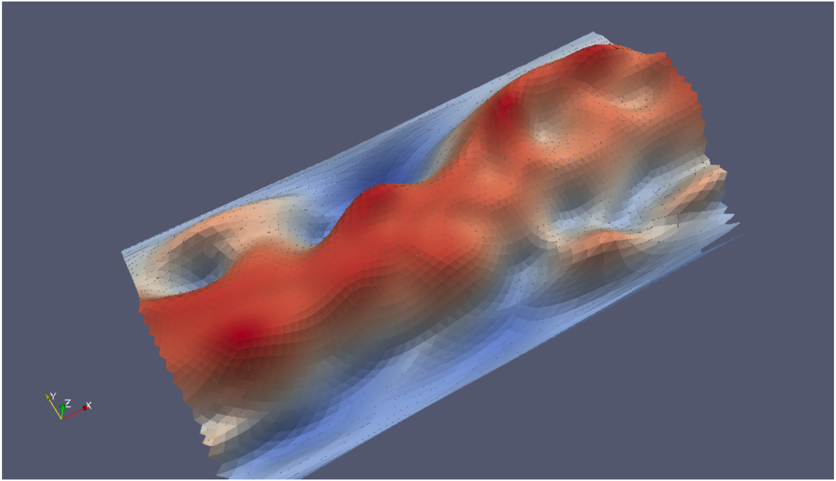

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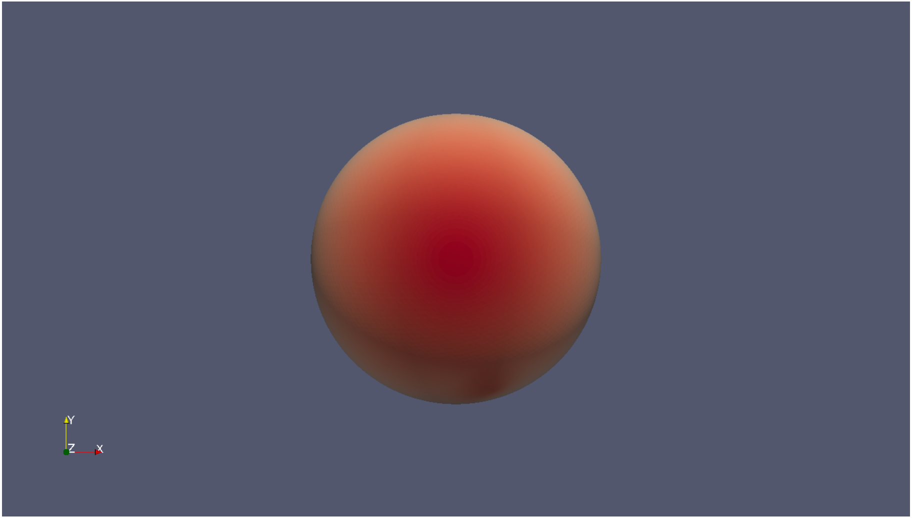

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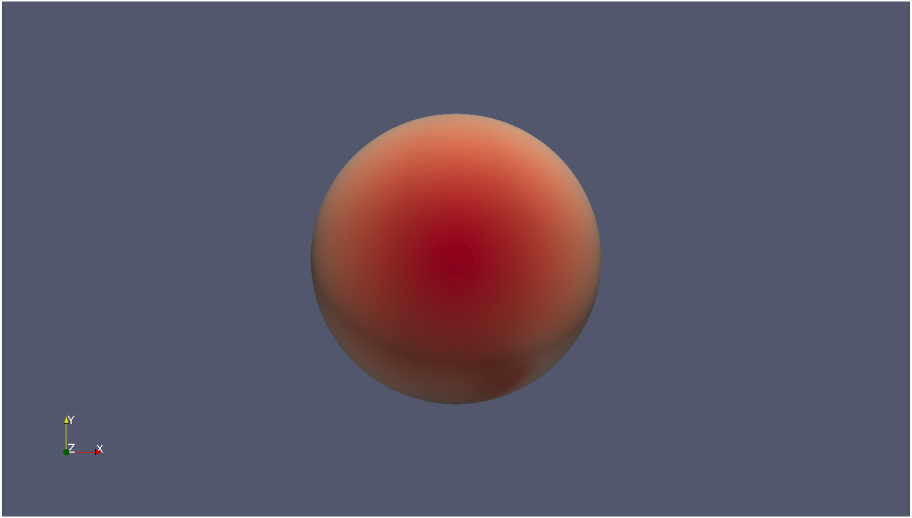

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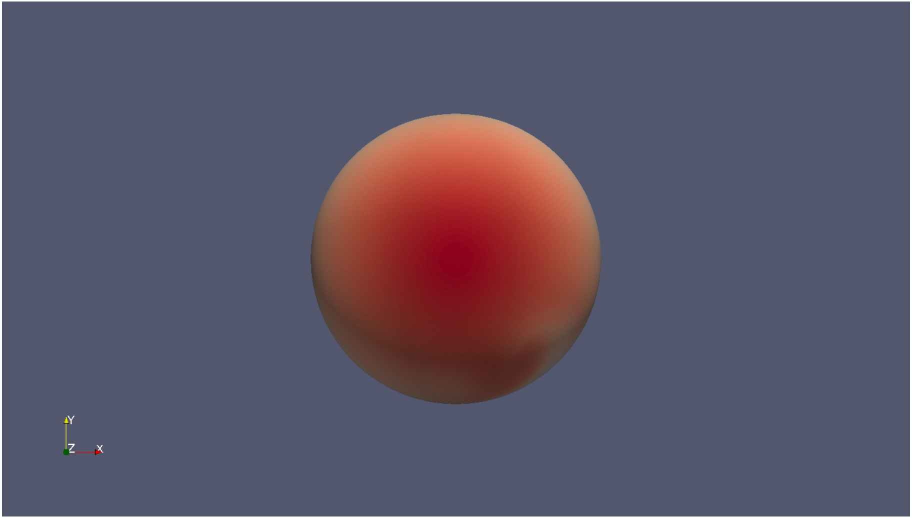

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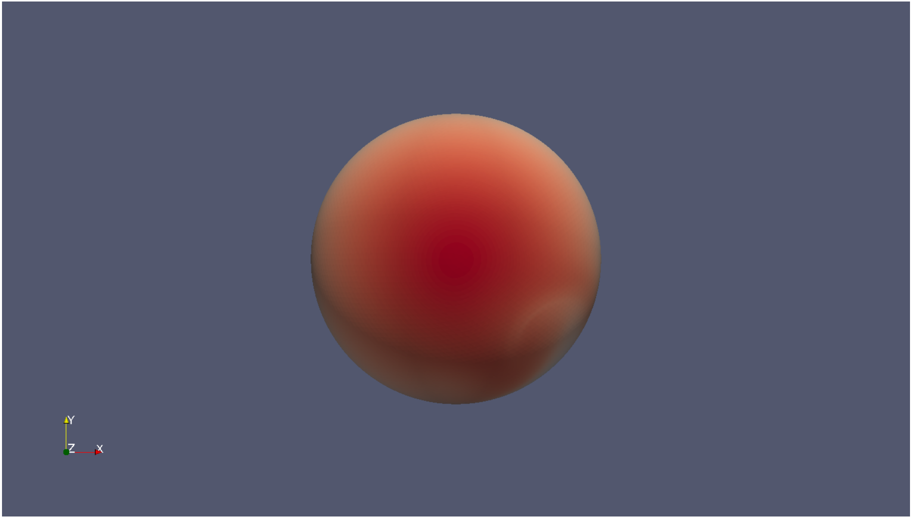

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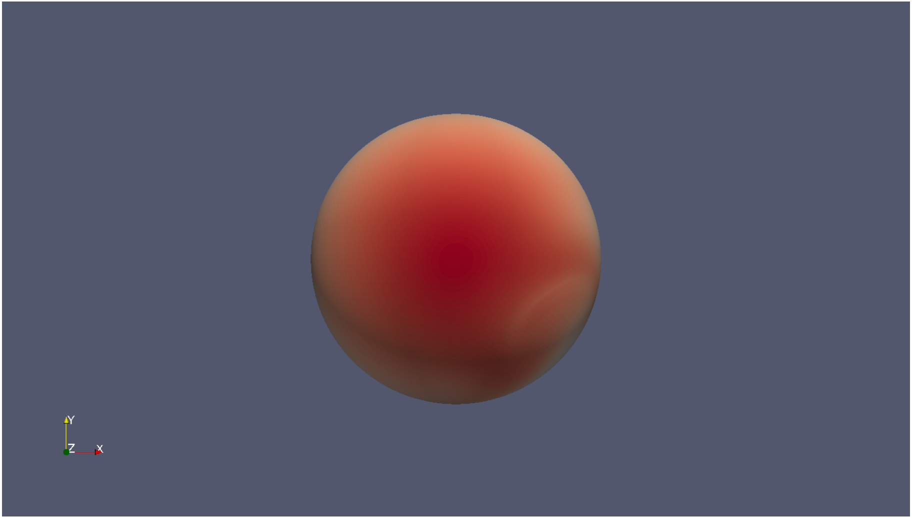

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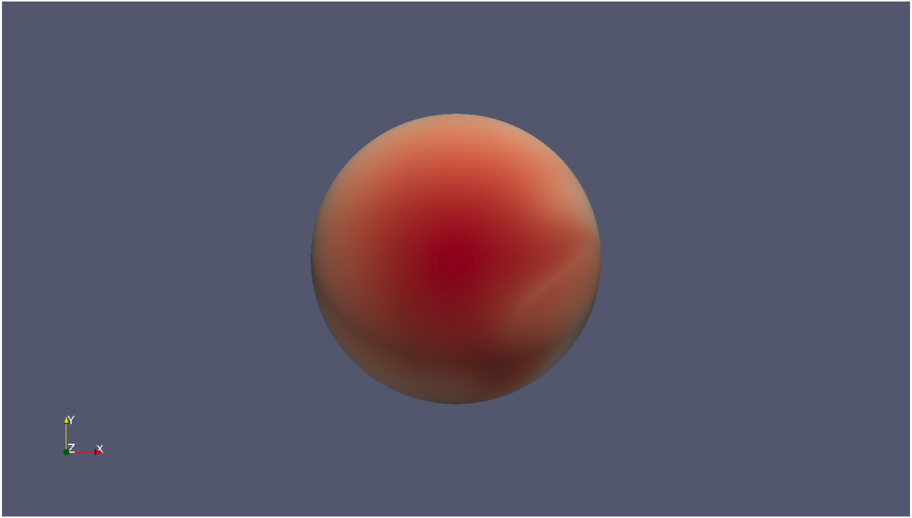

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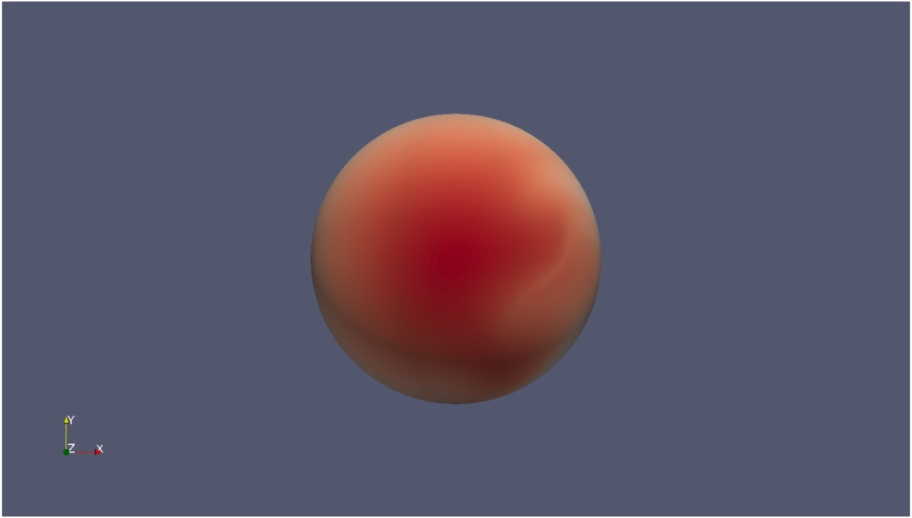
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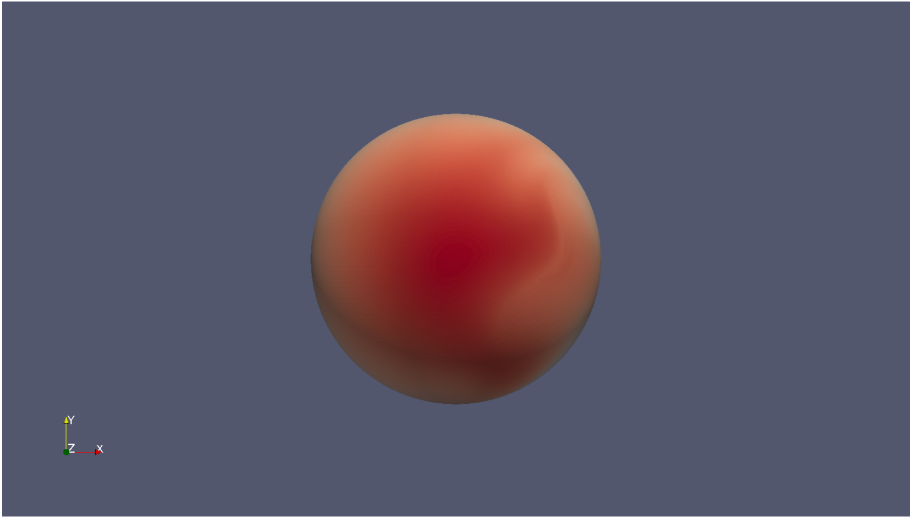

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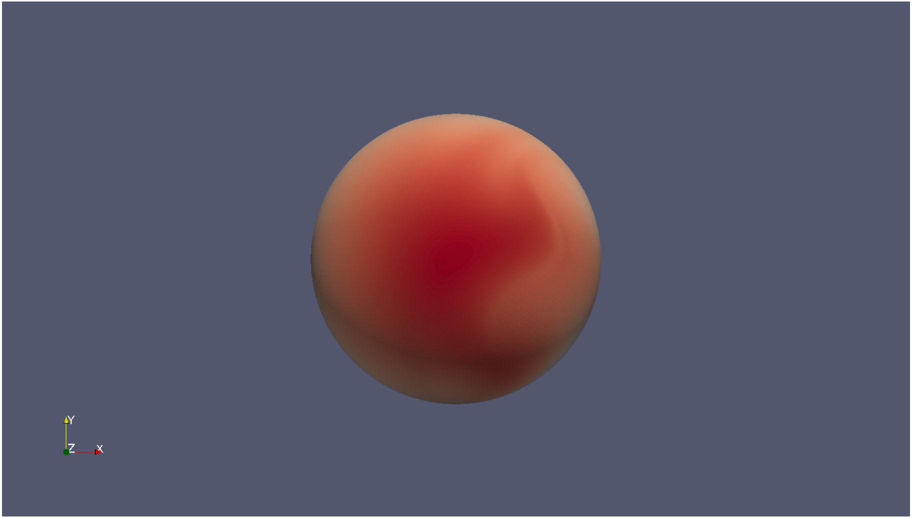

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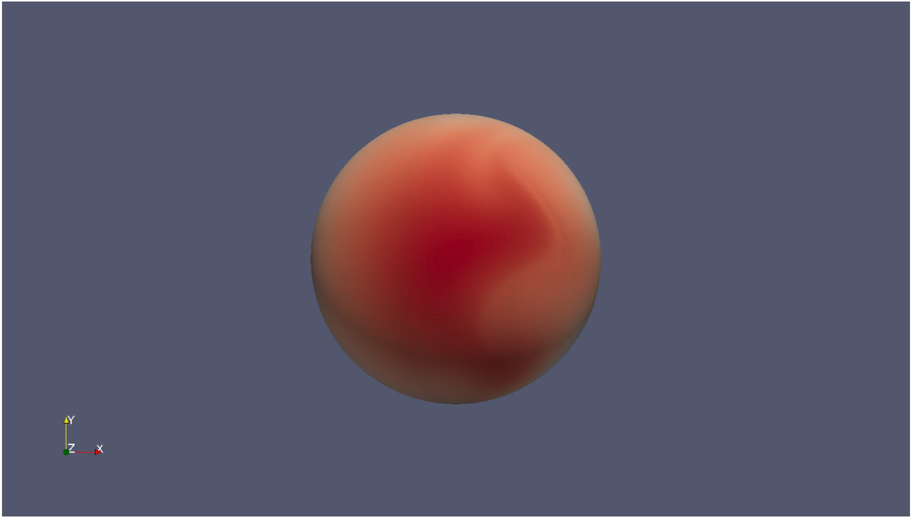

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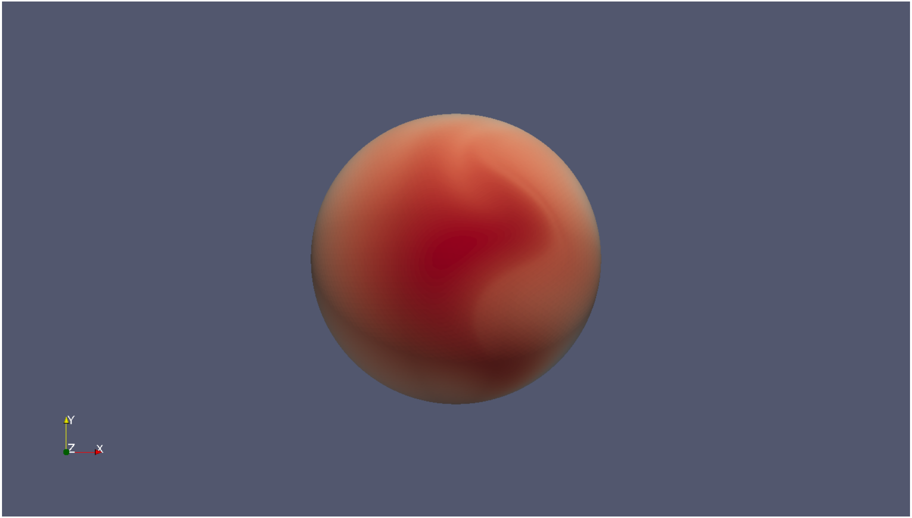

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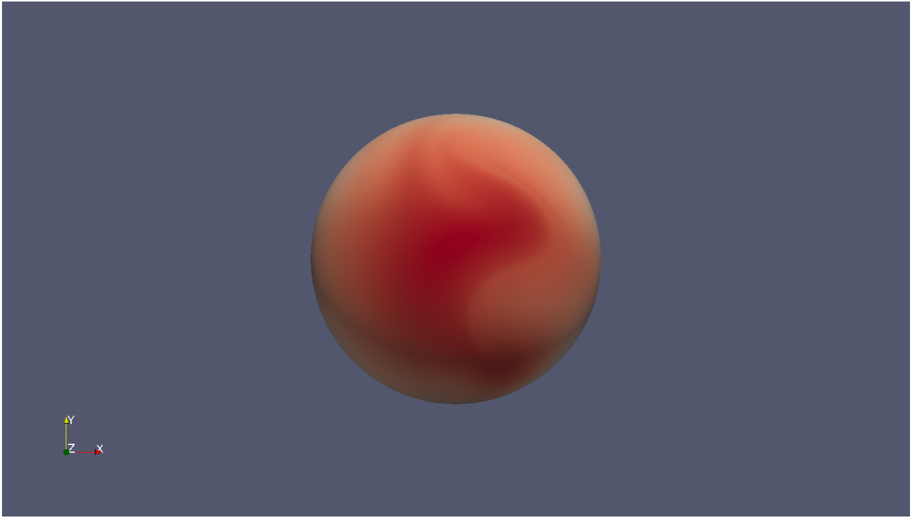

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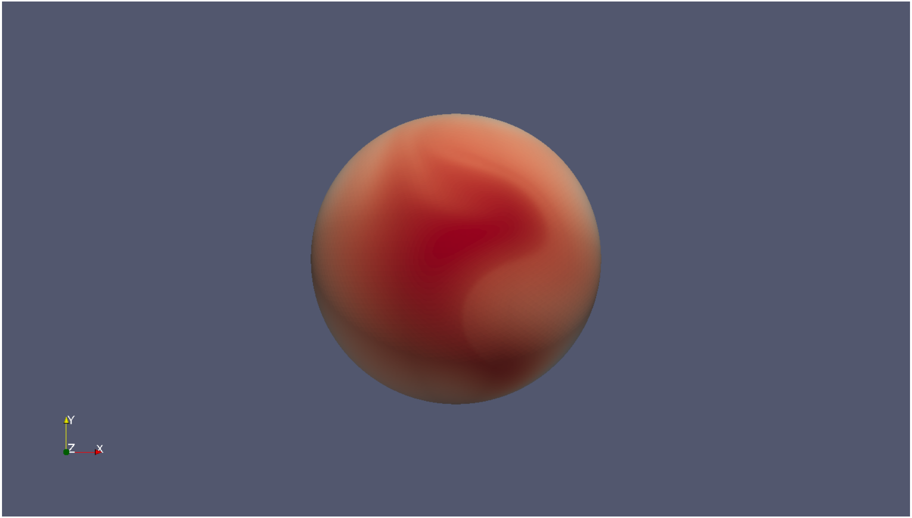

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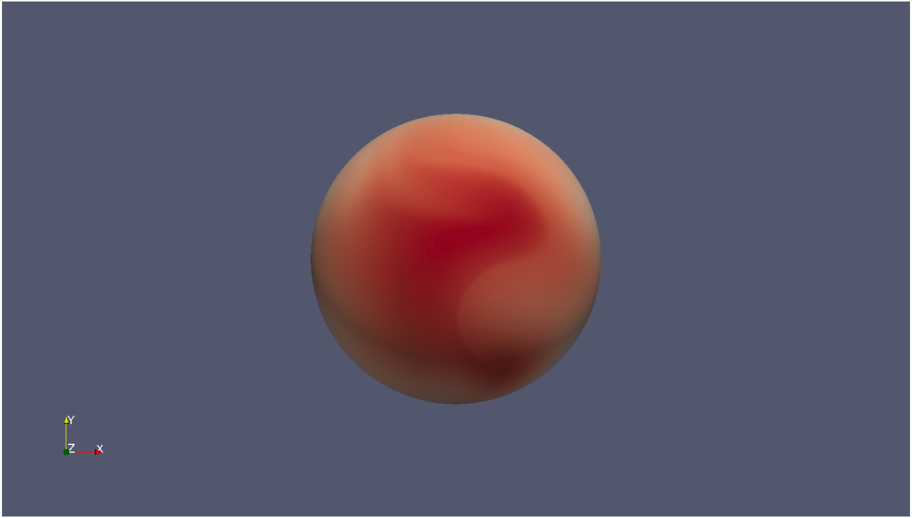

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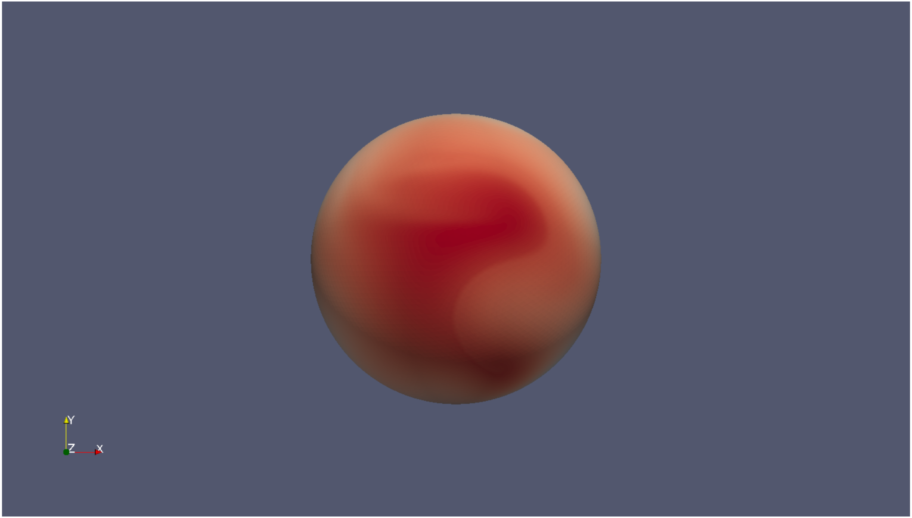

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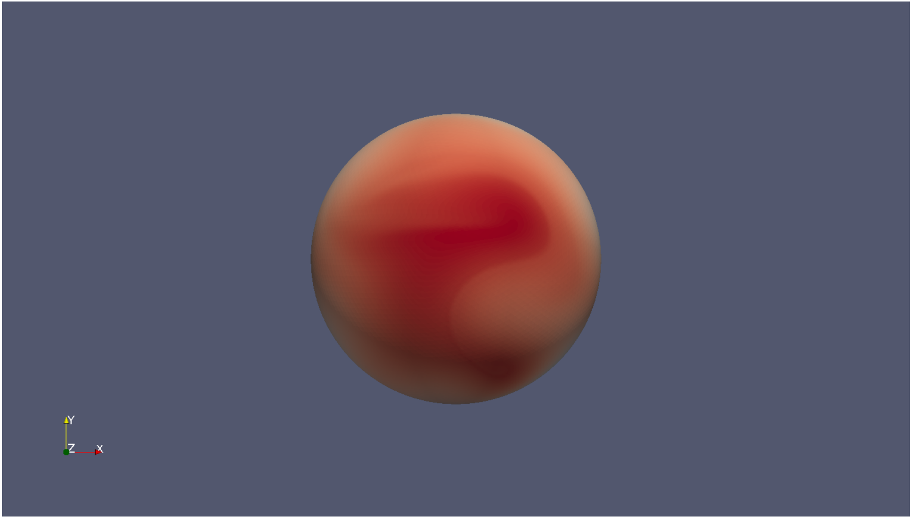

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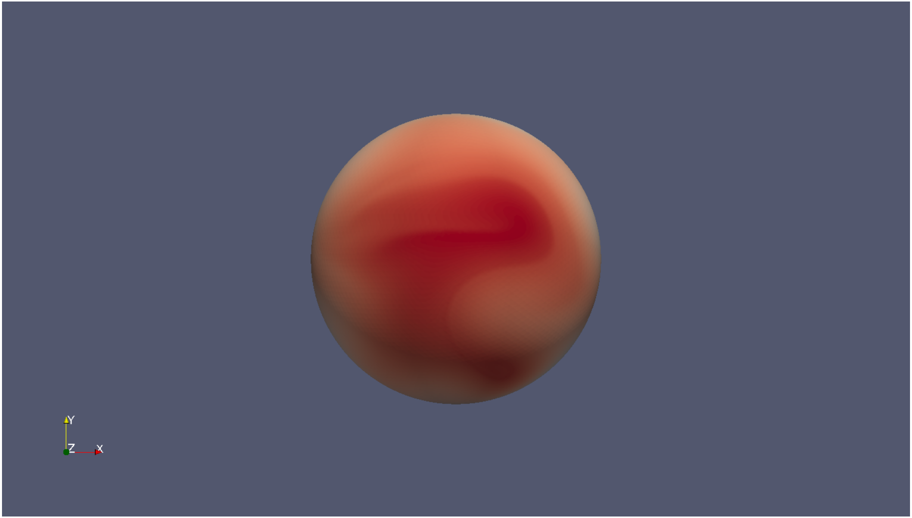

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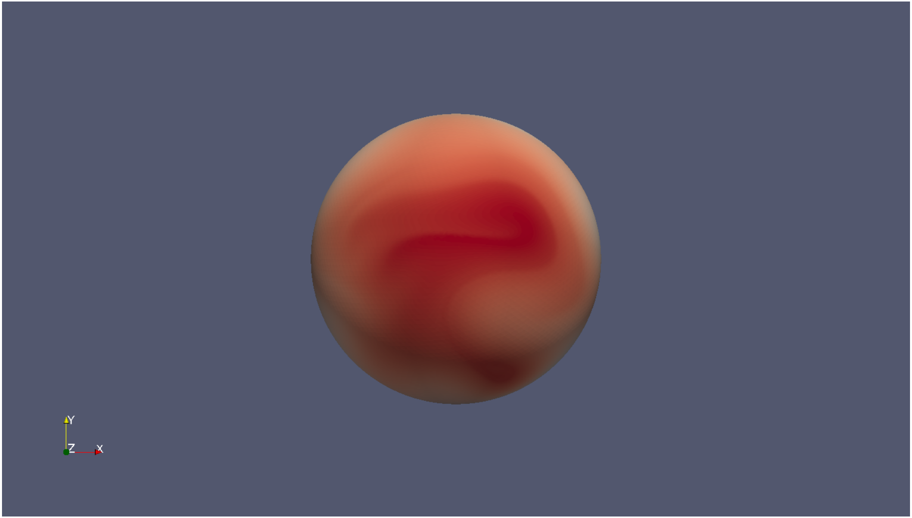

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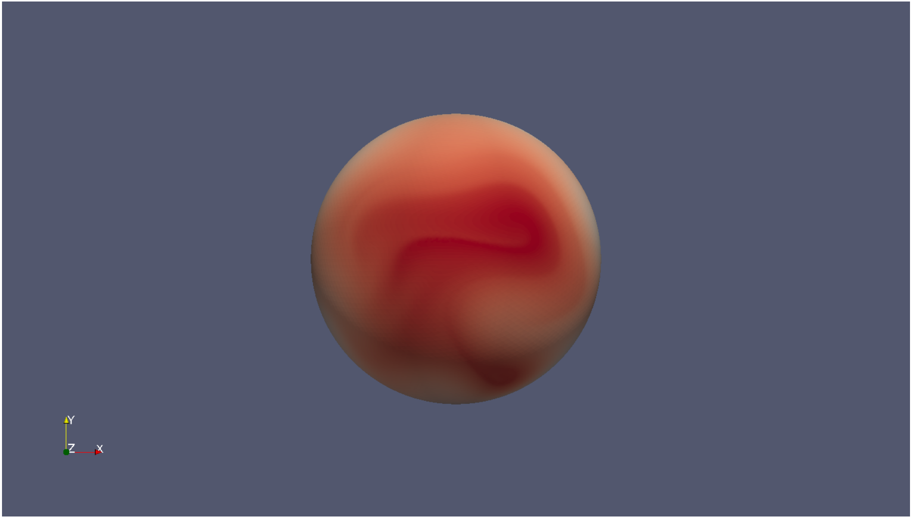

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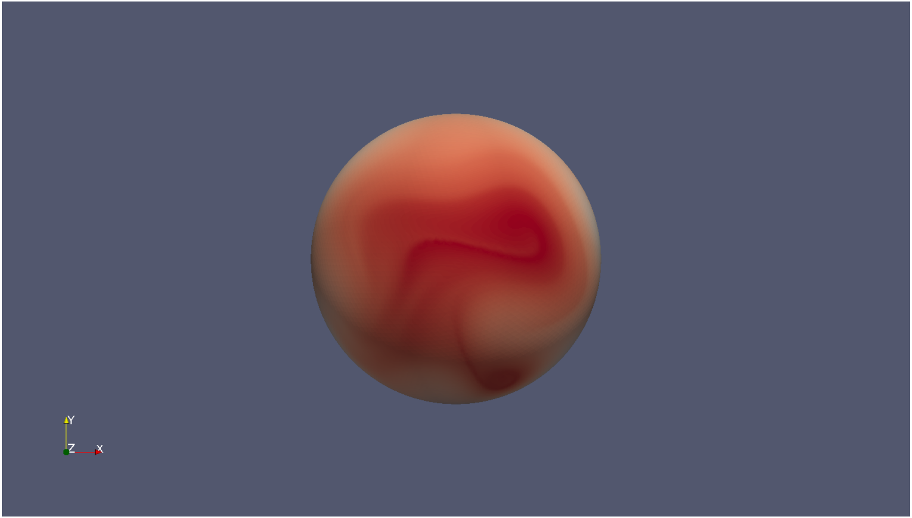

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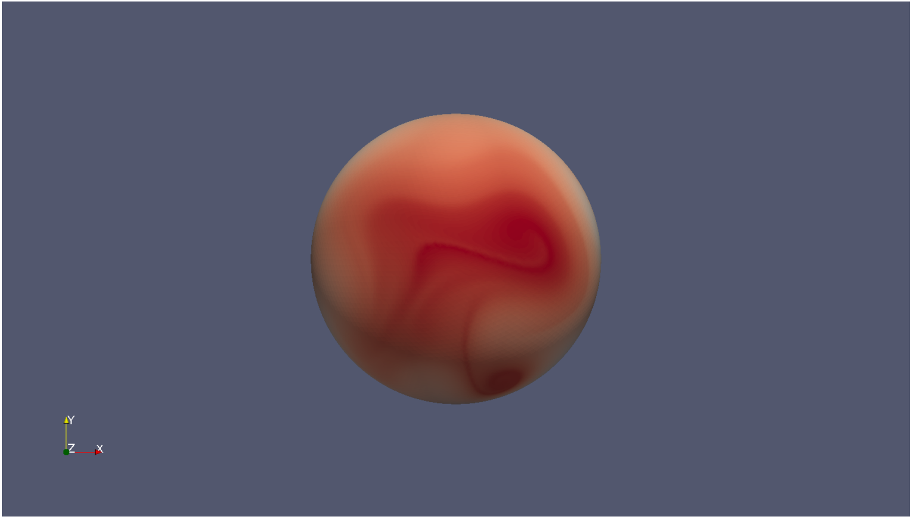

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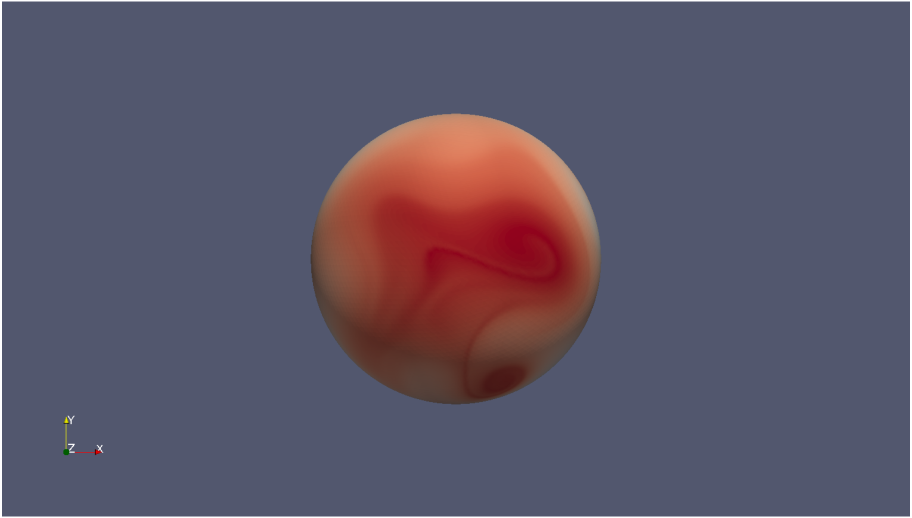

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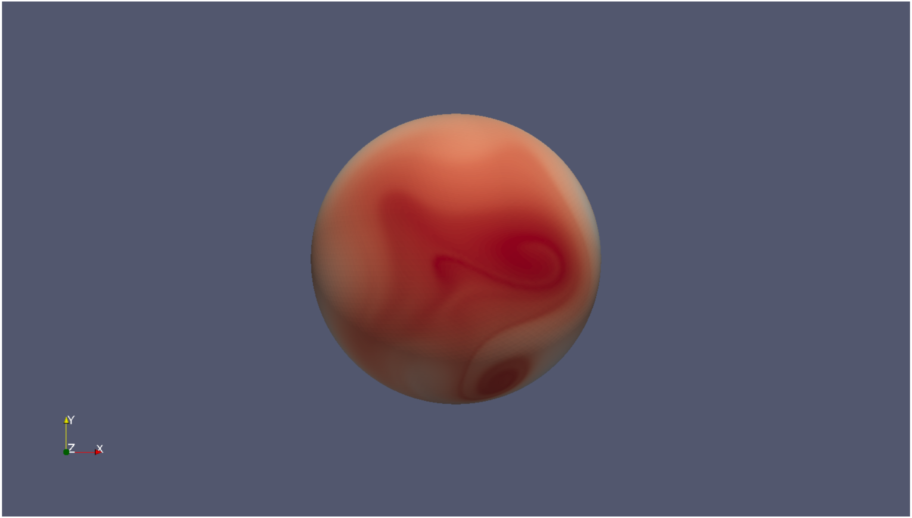

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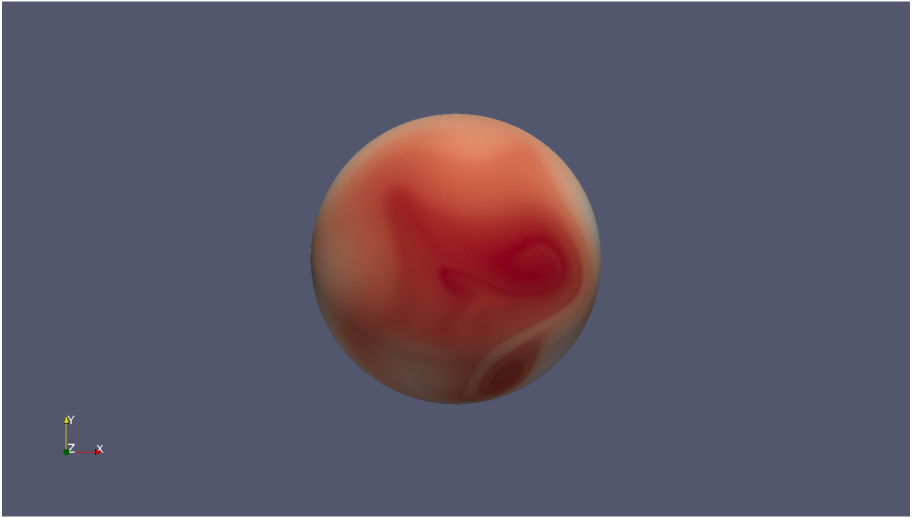

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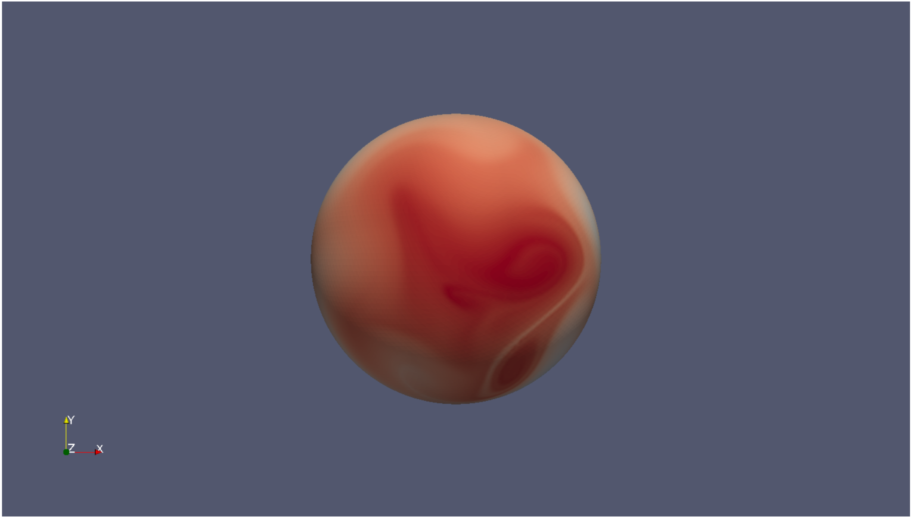

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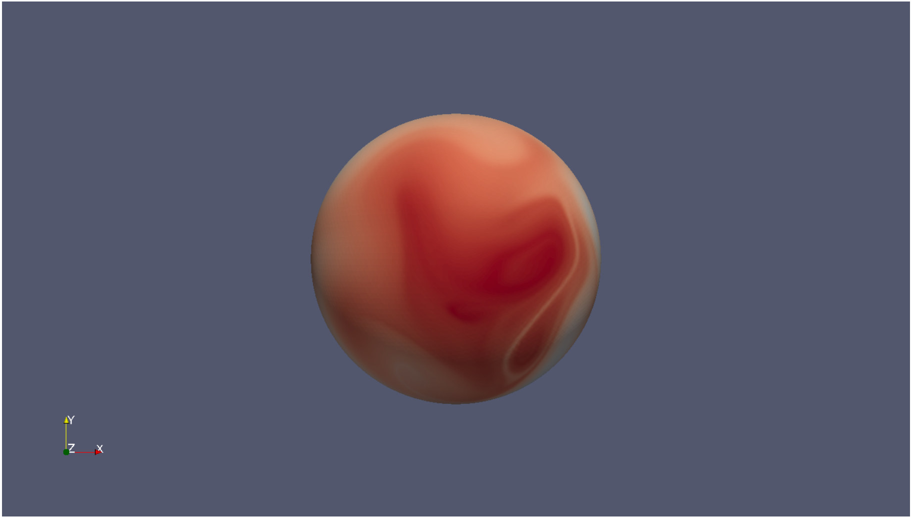

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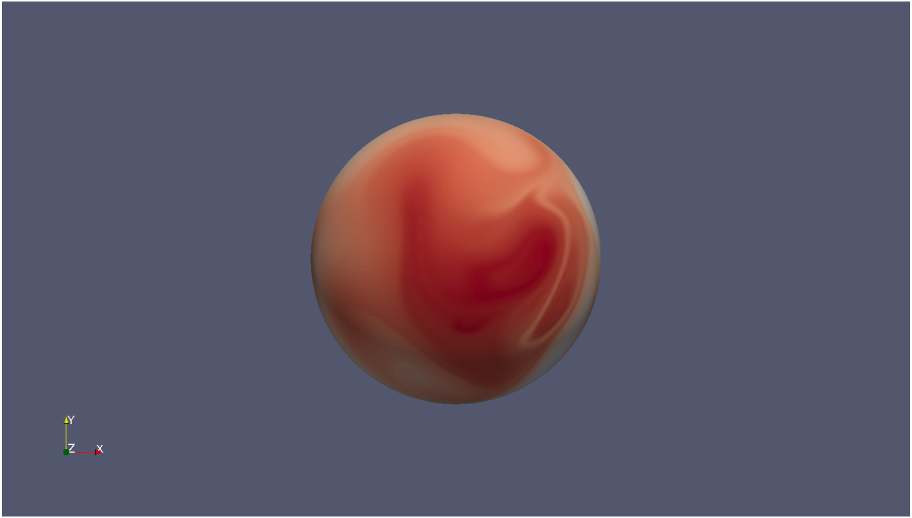

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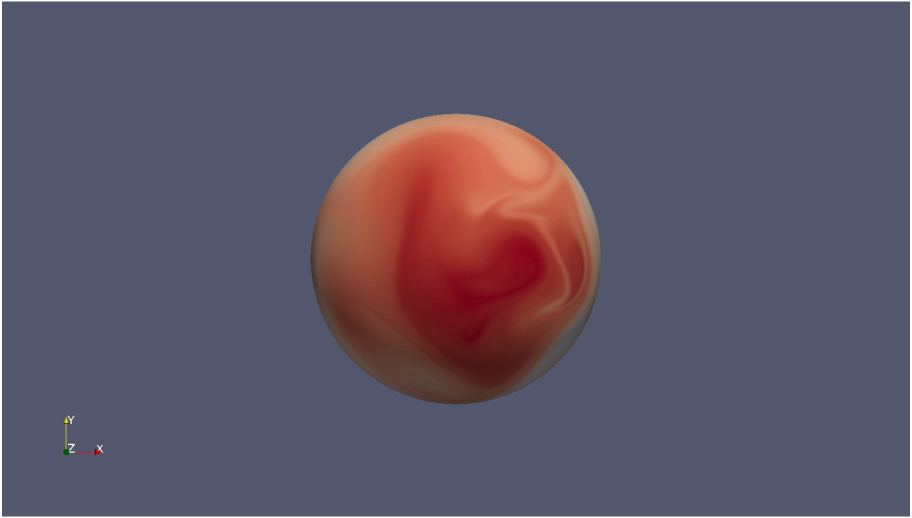

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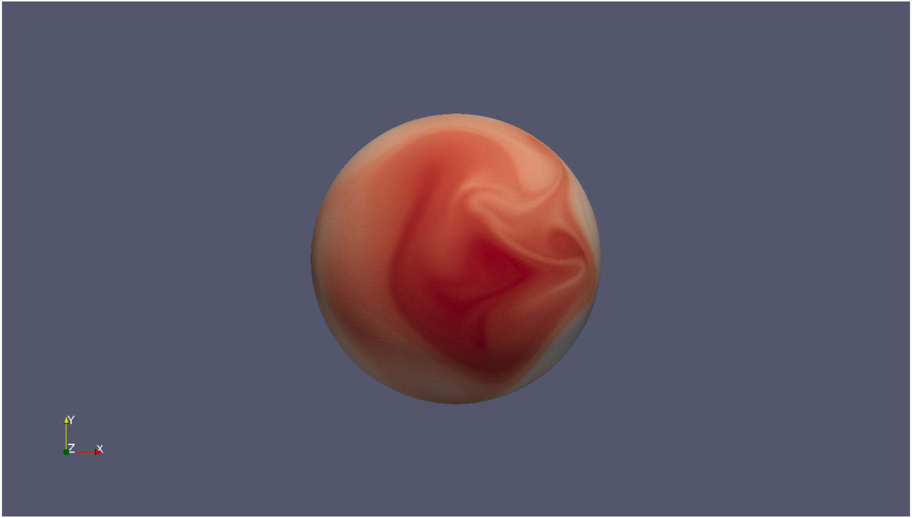

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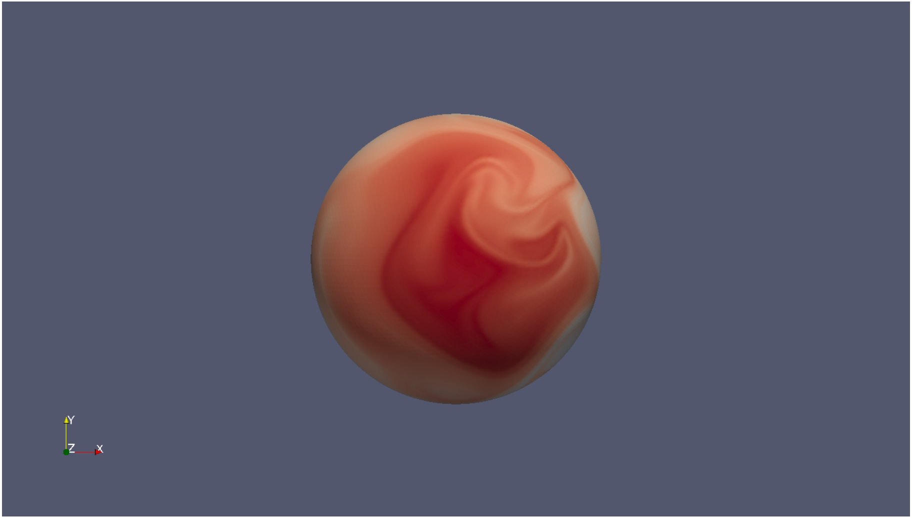

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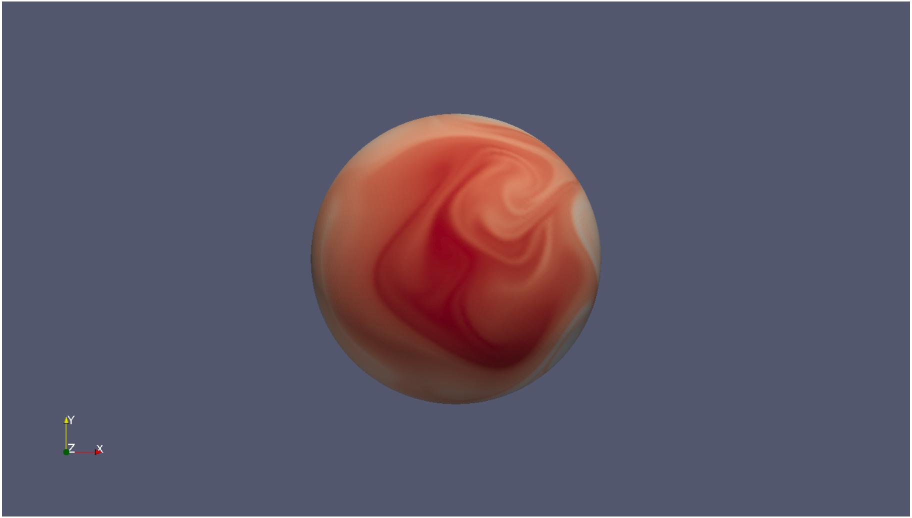
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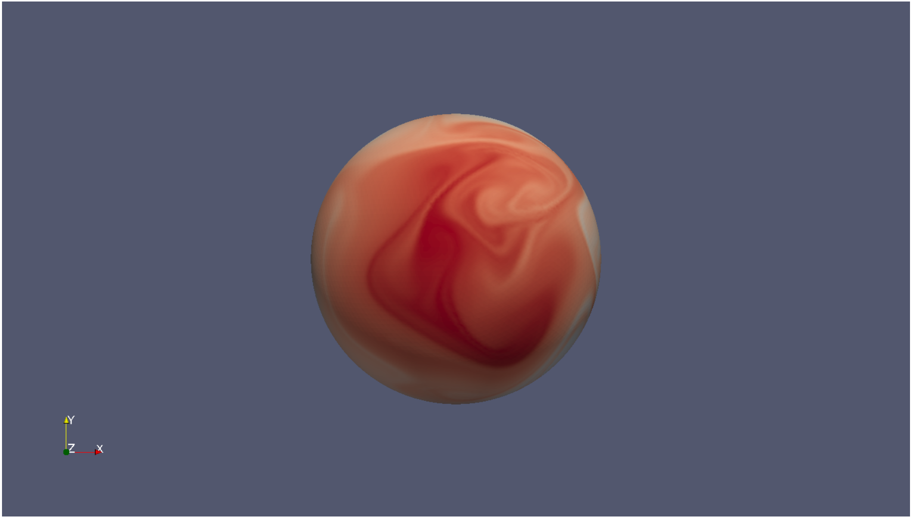

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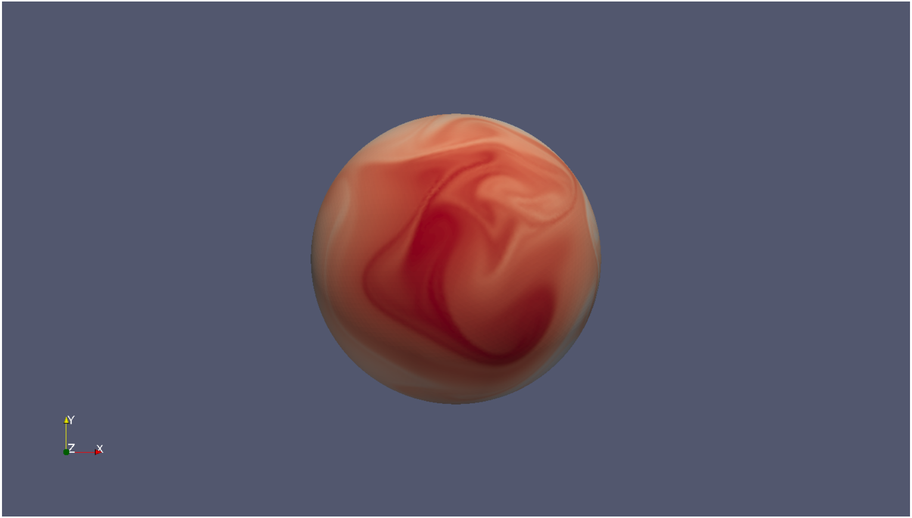

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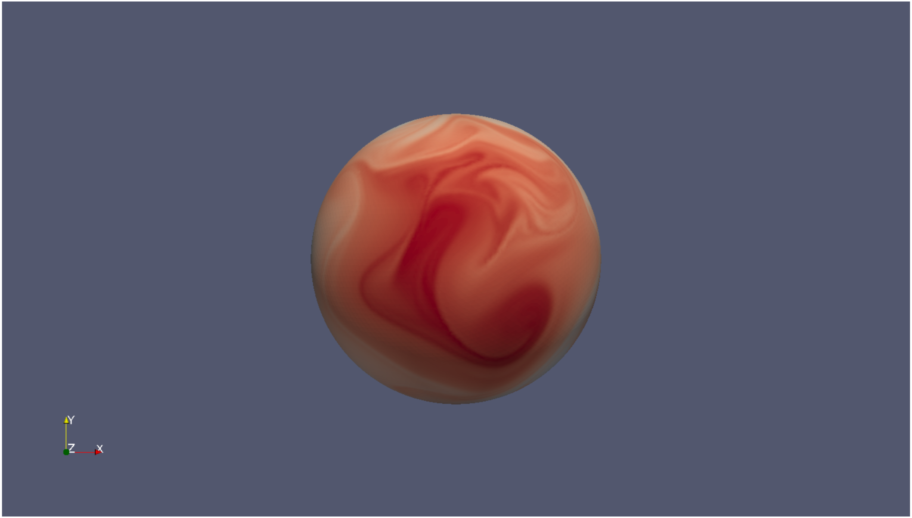

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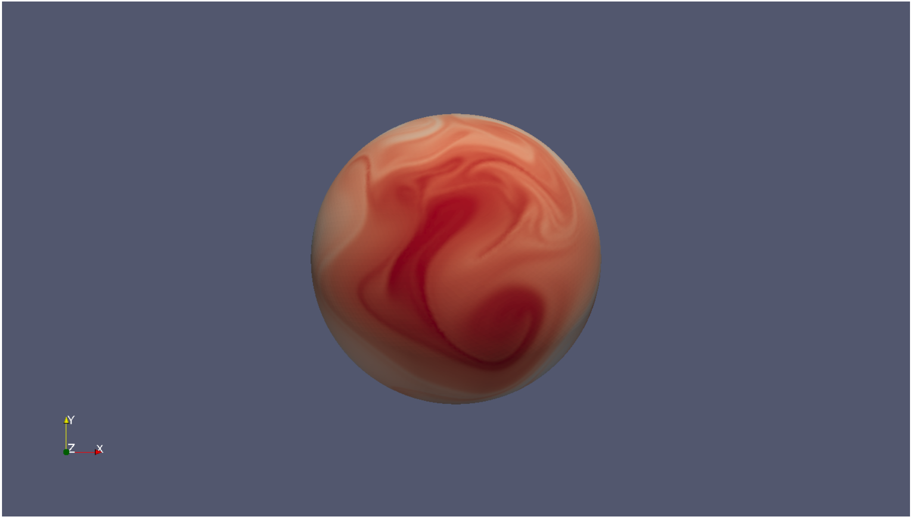

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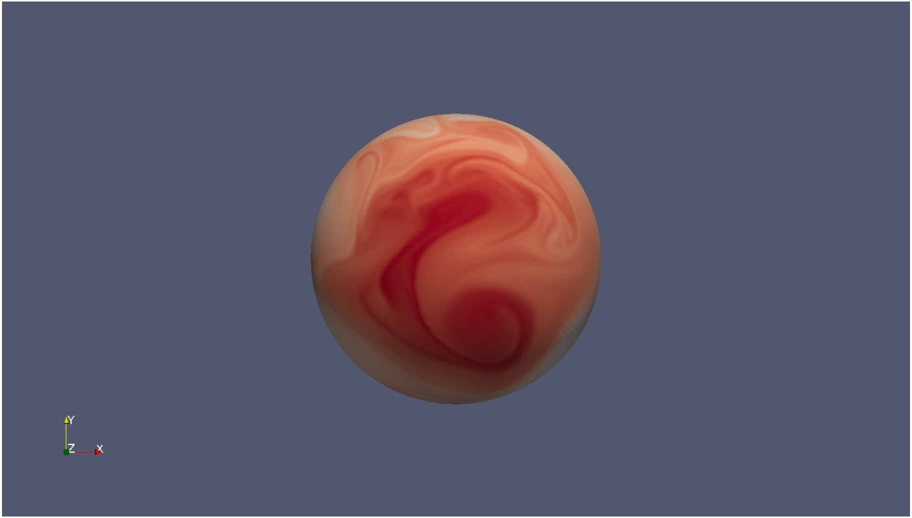

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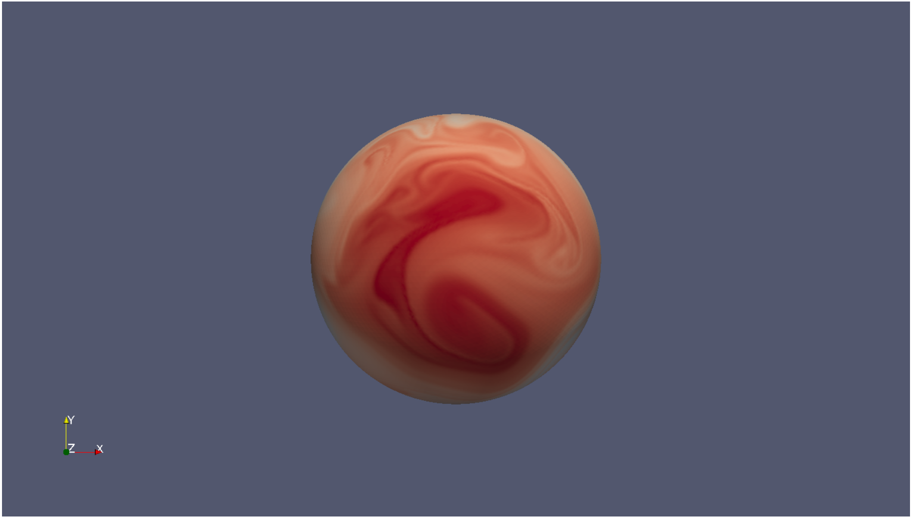

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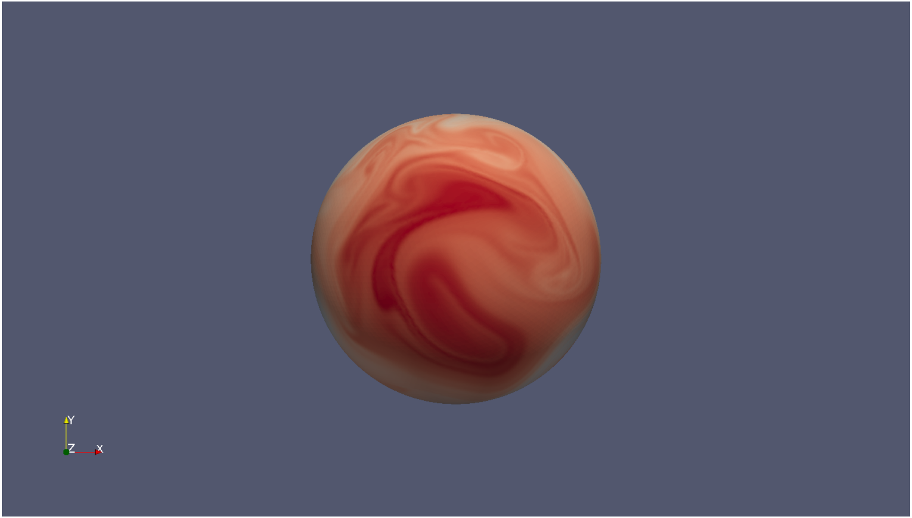
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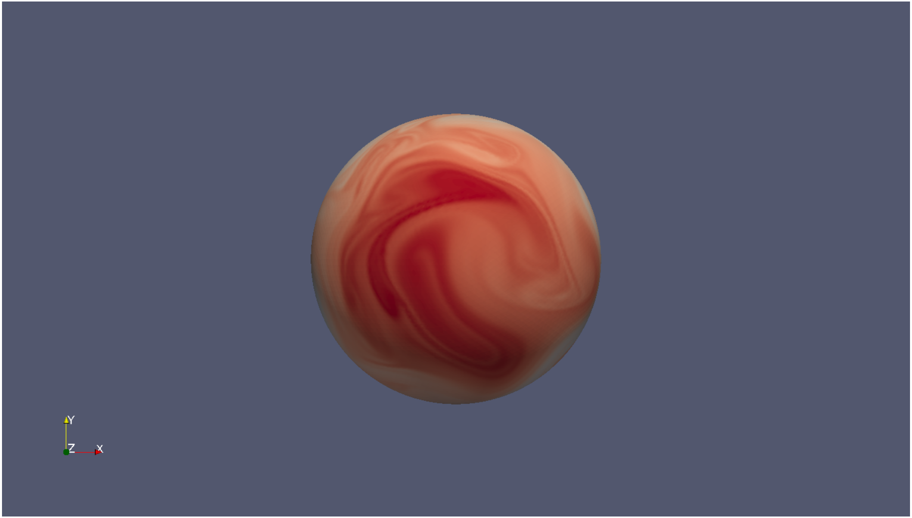

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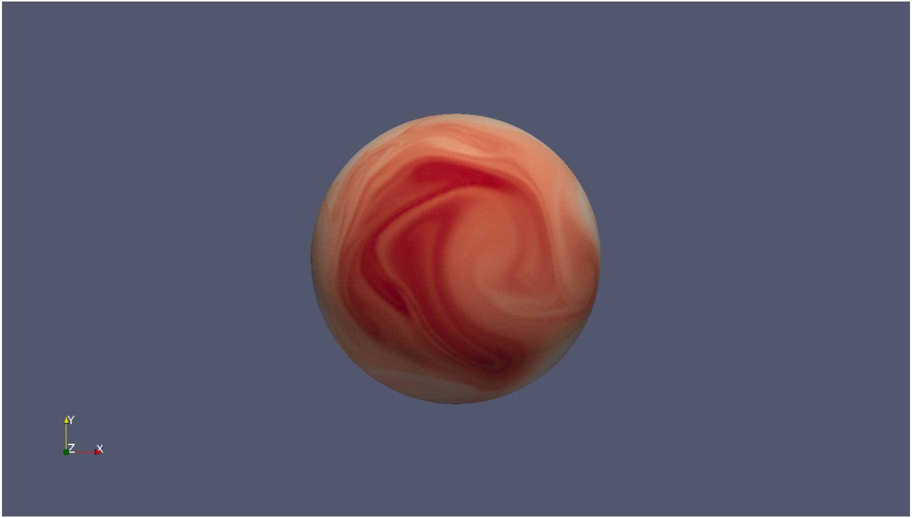

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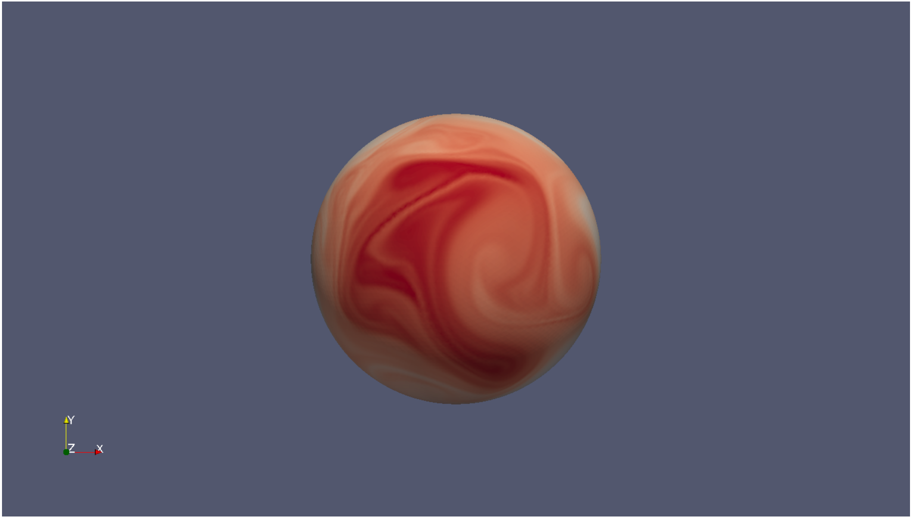

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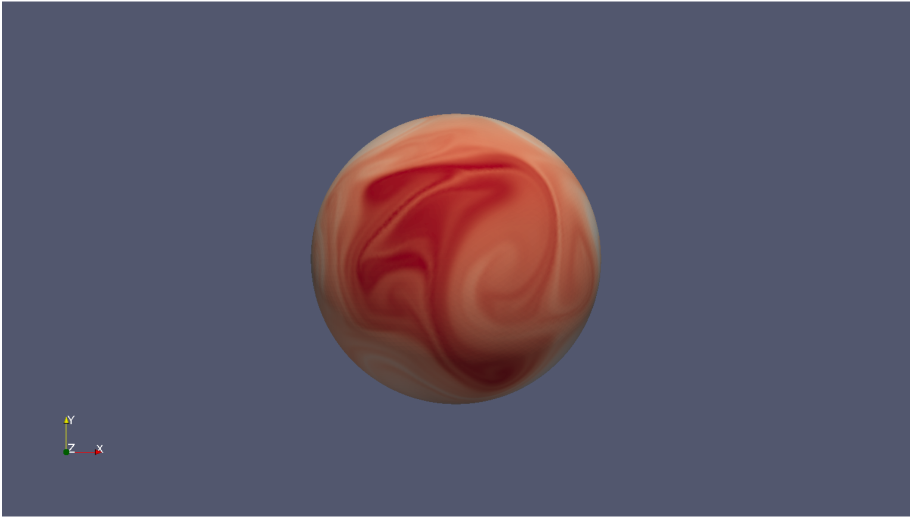

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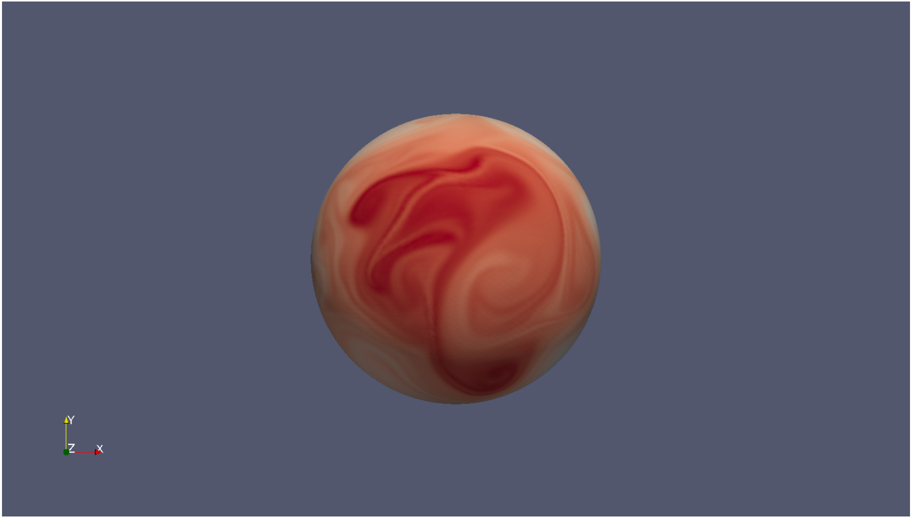

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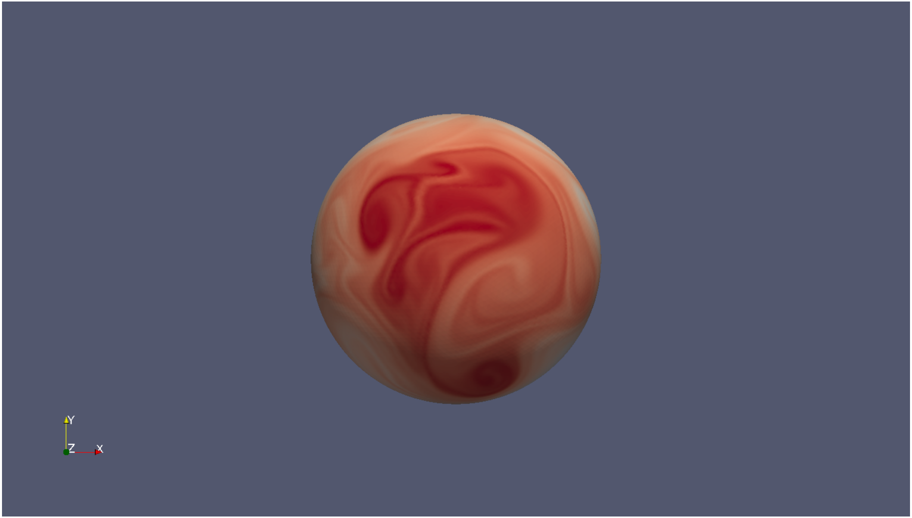

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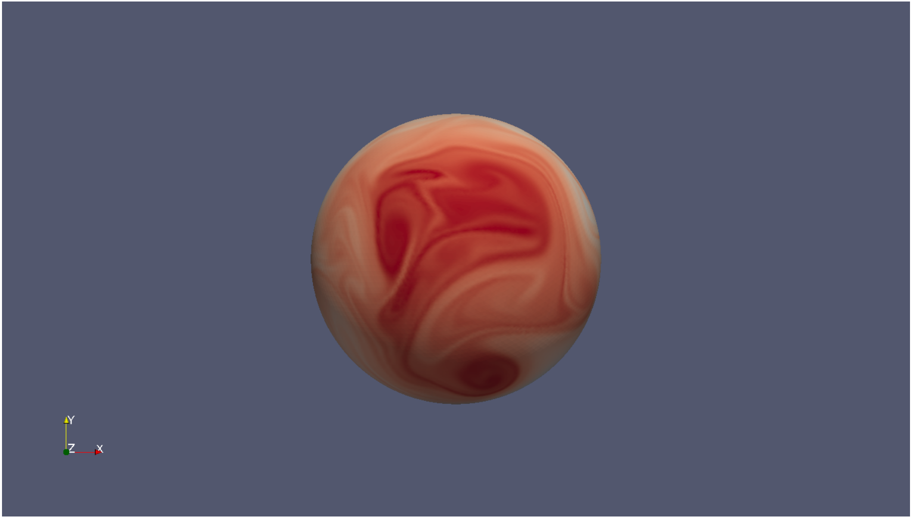

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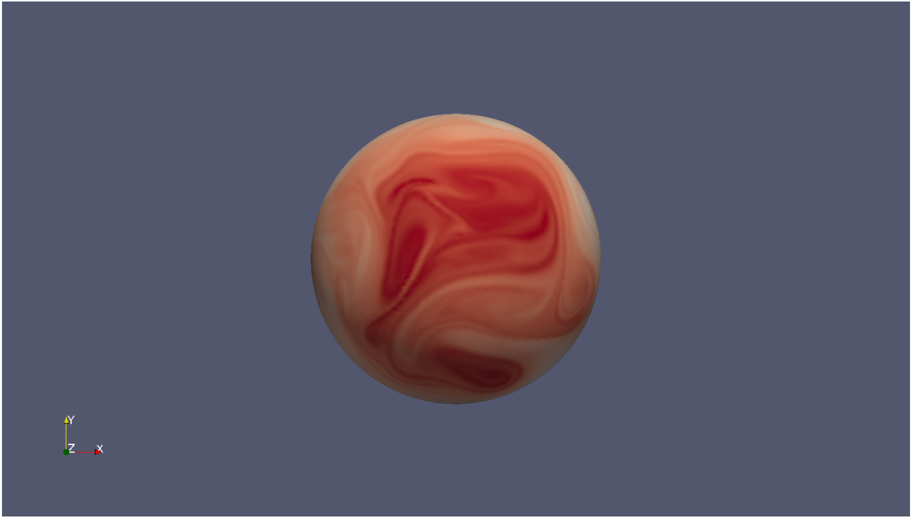

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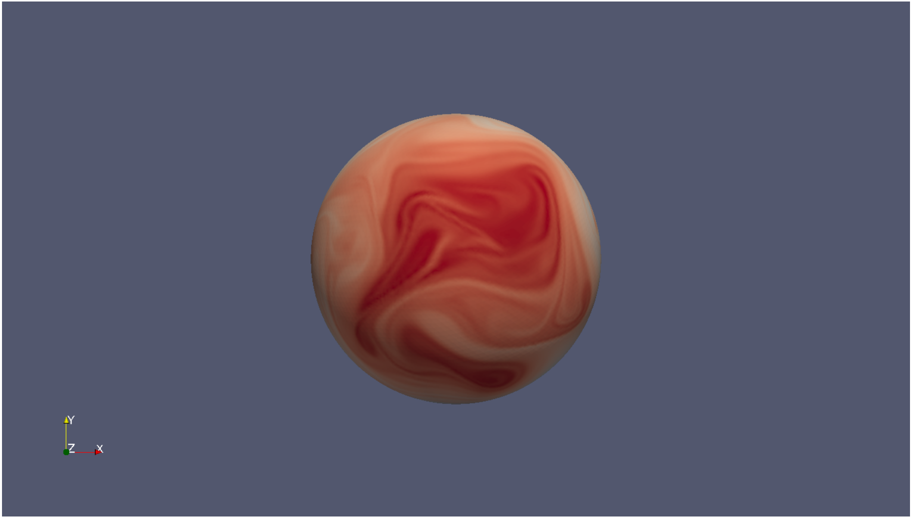

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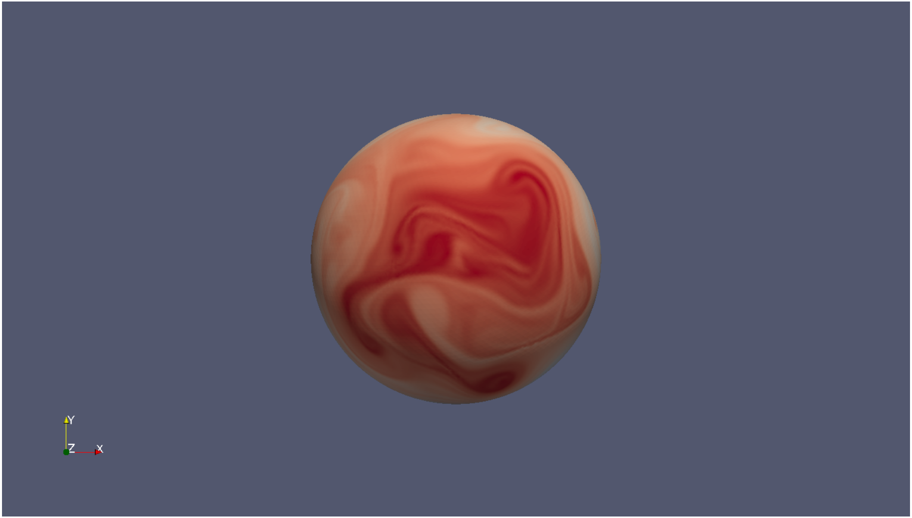

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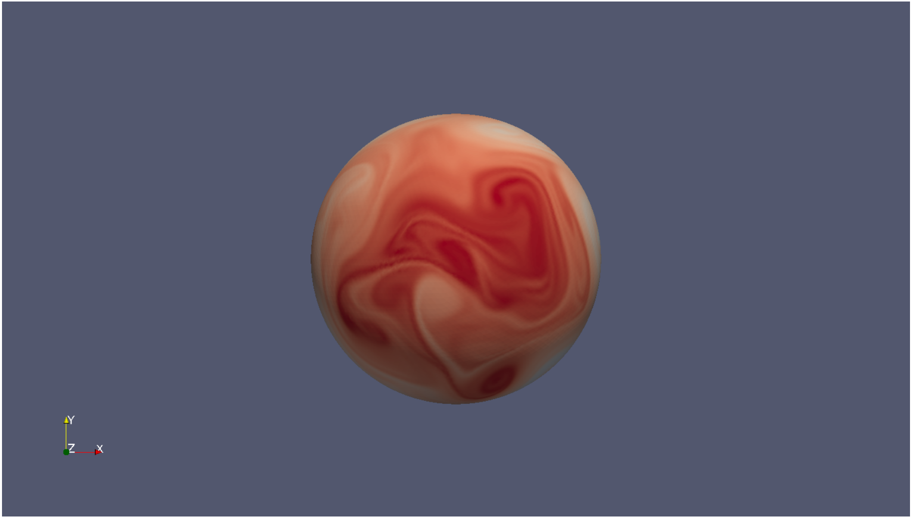
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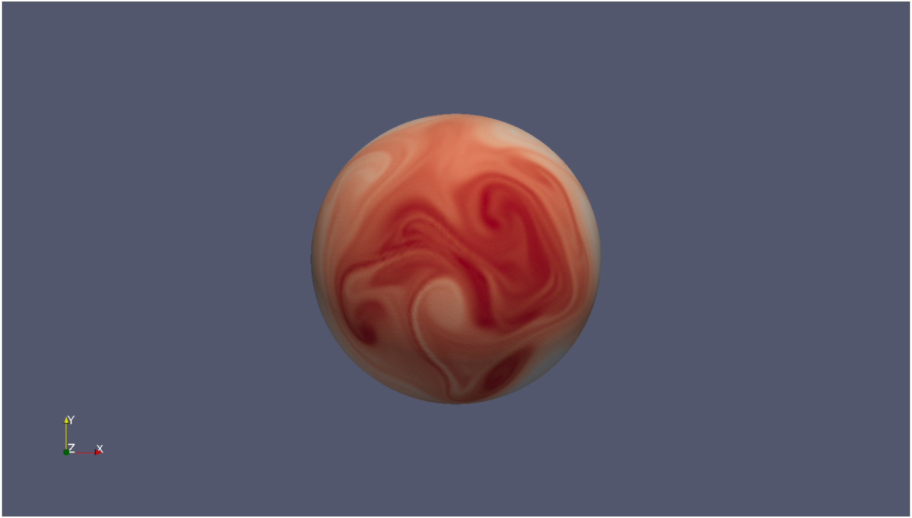

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Another story: dual meshes

$$\begin{tikzcd}
\mathbb{V}_d^2 \arrow[r, shift left=4, "\tilde{\nabla}^\perp"] \arrow[d, "\star_h"] & \mathbb{V}_d^1 \arrow[l, "\nabla^\perp \cdot"'] \arrow[r, shift left=4, "\tilde{\nabla}\cdot"] \arrow[d, "\star_h"] & \mathbb{V}_d^0 \arrow[l, "\nabla"'] \arrow[d, "\star_h"] \\
\mathbb{V}_p^0 \arrow[r, "\nabla^\perp"] & \mathbb{V}_p^1 \arrow[r, "\nabla\cdot"] \arrow[l, shift left=4, "\tilde{\nabla}^\perp \cdot"] & \mathbb{V}_p^2 \arrow[l, shift left=4, "\tilde{\nabla}"]
\end{tikzcd}$$

- J. Thuburn and CJC, *A framework for mimetic discretization of the rotating shallow-water equations on arbitrary polygonal grids*, SISC (2012).
- CJC and J. Thuburn, *A finite element exterior calculus framework for the rotating shallow-water equations*, (submitted to JCP, preprint on arXiv).

Towards 3D

$$\underbrace{\mathbb{W}_0}_{\text{cts},q?} \xrightarrow{\nabla} \underbrace{\mathbb{W}_1}_{\text{cts tangent cpts},\boldsymbol{\omega}} \xrightarrow{\nabla\times} \underbrace{\mathbb{W}_2}_{\text{cts normal cpts},\boldsymbol{u},\theta\hat{\boldsymbol{z}}} \xrightarrow{\nabla\cdot} \underbrace{\mathbb{W}_3}_{\text{discontinuous},\rho}$$

$$\xleftarrow{\tilde{\nabla}\cdot} \qquad \xleftarrow{\tilde{\nabla}\times} \qquad \xleftarrow{\tilde{\nabla}}$$

Conclusions

- Extends C-grid approach with flexibility to take a) higher-order, b) non-orthogonal grids, c) different DOF ratios.
- Mimetic finite elements/finite element exterior calculus based on sequence of FE spaces compatible with $\nabla^\perp$ and $\nabla\cdot$ to retain $\nabla \cdot \nabla^\perp = 0$.
- Dual operators $\tilde{\nabla}^\perp\cdot$ and $\tilde{\nabla}$ are defined weakly and satisfy $\tilde{\nabla}^\perp \cdot \tilde{\nabla}$.
- Steady geostrophic modes and diagnostic PV conservation.
- Energy/enstrophy conservation possible.
- Efficient semi-implicit implementation with accurate advection is possible.

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- J. Thuburn and CJC, *A framework for mimetic discretization of the rotating shallow-water equations on arbitrary polygonal grids*, SISC (2012).
- CJC and J. Shipton, *Mixed finite elements for numerical weather prediction*, JCP (2012).
- A. Staniforth, T. Melvin and CJC, *Analysis of a mixed finite-element pair proposed for an atmospheric dynamical core*, QJRMS (2012).
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- CJC and J. Thuburn, *A finite element exterior calculus framework for the rotating shallow-water equations*, (submitted to JCP, preprint on arXiv).
- M. Rognes, CJC, D. Ham and A. McRae, *Automating the solution of PDEs on the sphere and other manifolds* (submitted to GMD, viewable on GMDD).
- A. McRae and CJC, *Energy-enstrophy conserving mixed finite element schemes for the rotating shallow water equations* (submitted to QJRMS, preprint on arXiv).

3 year postdoctoral research associate position available!