Constructing a background error covariance model for variational ocean data assimilation

…*with an emphasis on the tropics*

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Acknowledgments:

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Outline

- The variational assimilation problem.
- Some remarks about **B**.
- General approach to modelling **B** for the ocean:
  - modelling correlation functions
  - parametrising variances
  - including balance and conservation constraints
- Examples from 3D-Var and 4D-Var with the OPA OGCM.

The variational assimilation problem

Minimize $$J = J_b + J_o$$

Background term

$$J_b = \tfrac{1}{2}(\mathbf{x} - \mathbf{x}^b)^T \mathbf{B}^{-1} (\mathbf{x} - \mathbf{x}^b)$$

Observation term

$$J_o = \tfrac{1}{2}\left(G(\mathbf{x}) - \mathbf{y}^o\right)^T \mathbf{R}^{-1} \left(G(\mathbf{x}) - \mathbf{y}^o\right)$$

The incremental approximation

Minimize $$J = J_b + J_o$$

Background term

$$J_b = \tfrac{1}{2}\,\delta\mathbf{x}^T\,\mathbf{B}^{-1}\,\delta\mathbf{x}$$

where $\delta\mathbf{x} = \mathbf{x} - \mathbf{x}^b$

Observation term (quadratic)

$$J_o = \tfrac{1}{2}\left(\mathbf{G}\delta\mathbf{x} - \mathbf{d}\right)^T \mathbf{R}^{-1}\left(\mathbf{G}\delta\mathbf{x} - \mathbf{d}\right)$$

where $\mathbf{d} = \mathbf{y}^o - G\left(\mathbf{x}^b\right)$

Preconditioning with **B**

Background term

$$J_b = \tfrac{1}{2}\,\delta\mathbf{x}^T\,\mathbf{B}^{-1}\,\delta\mathbf{x}$$

Preconditioned background term

Define $\quad \mathbf{v} = \mathbf{U}^{-I}\,\delta\mathbf{x}$

where $\quad \mathbf{B}^{-1} = \left(\mathbf{U}^{-I}\right)^T \left(\mathbf{U}^{-I}\right)$ and $\mathbf{B} = \mathbf{U}\,\mathbf{U}^T$

then $\quad J_b = \tfrac{1}{2}\,\mathbf{v}^T\,\mathbf{v}$

Preconditioning with **B**

- $\mathbf{v}$ is the control vector for the minimization problem.
- On the first inner iteration we take $\delta\mathbf{x} = 0$ so $\mathbf{v} = 0$.
- Consequently, on each inner iteration, we only need to specify the inverse of the change of variable:

$$\delta\mathbf{x} = \mathbf{U}\,\mathbf{v}$$

and its adjoint for computing the gradient of J_o

$$\mathbf{v}^* = \mathbf{U}^T\,\delta\mathbf{x}^*$$

Some general remarks about **B**

- **B** largely determines how observational increments are smoothed in space and transferred between different model variables.

Linear solution: $$\delta \mathbf{x}^a = \mathbf{B}\underbrace{\mathbf{G}^T\left(\mathbf{G}\mathbf{B}\mathbf{G}^T + \mathbf{R}\right)^{-1}\mathbf{d}}_{\hat{\mathbf{x}}}$$

- **B** is important in both 3D-Var and 4D-Var.

- **B** is also important for ensemble methods for generating realistic initial perturbations (using the square root factor **U**).

Some general remarks about **B**

- Difficulty diagnosing statistics: there is not enough (and never will be) enough information to determine all the elements of **B** (typically > $O(10^{11})$).
- Computational difficulty: **B** is too large to store as a full matrix.
- **B** must be approximated using a model.
- In 3D-Var/4D-Var, **B** must be implemented as an operator :

$$\hat{\mathbf{x}} \xrightarrow{\mathbf{B}} \mathbf{x}$$

unless the analysis space is sufficiently small (e.g., coefficients of a few ensemble members or EOFs).

Some general remarks about **B**

- Constructing an effective **B** model involves substantial development and tuning!

Some specific remarks about **B** for the ocean

- The background state variables in an OGCM:
  - temperature (*T*), salinity (*S*), sea-surface height (*SSH*), horizontal velocity (*u, v*).

    (e.g., Weaver et al. 2003 - MWR; Vialard et al. 2003 - MWR)

  but may also include the surface forcing fields:
  - wind stress (*taux, tauy*), heat flux (*Q*), evaporation-precipitation (*E-P*).

    (e.g., Bonekamp et al. 2001 – JGR)

- Ocean observations are relatively sparse so it is difficult to estimate background error statistics from innovations. Considerable spatial and temporal averaging is required *(e.g., Martin et al. 2002).*

Some specific remarks about **B** for the ocean

- With few observations the role of **B** is critical for exploiting the available data-sets effectively (e.g., surface altimeter data).

- Added complexity due to the presence of continental boundaries (natural inhomegeneity, boundary conditions, scales, spectra, balance).

- Rich variety of scales: mesoscale (Gulf Stream, Kuroshio regions) ~O(10km) and synoptic scale (tropics) ~ O(100km). *(e.g., see Martin et al. 2002)*

A general approach for modelling **B**

- By definition,

$$\mathbf{B} = \begin{pmatrix} E\left[T' T'^T\right] & E\left[T' S'^T\right] & E\left[T' \eta'^T\right] & E\left[T' u'^T\right] & E\left[T' v'^T\right] \\ E\left[S' T'^T\right] & E\left[S' S'^T\right] & E\left[S' \eta'^T\right] & E\left[S' u'^T\right] & E\left[S' v'^T\right] \\ E\left[\eta' T'^T\right] & E\left[\eta' S'^T\right] & E\left[\eta' \eta'^T\right] & E\left[\eta' u'^T\right] & E\left[\eta' v'^T\right] \\ E\left[u' T'^T\right] & E\left[u' S'^T\right] & E\left[u' \eta'^T\right] & E\left[u' u'^T\right] & E\left[u' v'^T\right] \\ E\left[v' T'^T\right] & E\left[v' S'^T\right] & E\left[v' \eta'^T\right] & E\left[v' u'^T\right] & E\left[v' v'^T\right] \end{pmatrix}$$

where T', S' etc. denote the difference between the background and “true” values of the state variables (assumed unbiased).

A general approach for modelling **B**

(cf. Derber & Bouttier 1999 - Tellus)

- Suppose (to be justified shortly)

"balanced" variables

"unbalanced" variables

$$
\begin{aligned}
T' &= T'_B \\
S' &= S'_B + S'_U = \underbrace{K_{ST}T'}_{S'_B} + S'_U \\
\eta' &= \eta'_B + \eta'_U = \underbrace{K_{\eta T}T' + K_{\eta S}S'}_{\eta'_B} + \eta'_U \\
u' &= u'_B + u'_U = \underbrace{K_{uT}T' + K_{uS}S' + K_{u\eta}\eta'}_{u'_B} + u'_U \\
v' &= v'_B + v'_U = \underbrace{K_{vT}T' + K_{vS}S' + K_{v\eta}\eta'}_{v'_B} + v'_U
\end{aligned}
$$

A general approach for modelling **B**

- Substitute the expressions for T', S', η', u', v' into the general expression for $\mathbf{B}$ and assume that $T'_B, S'_U, \eta'_U, u'_U, v'_U$ are mutually uncorrelated.

- Then we can write $\mathbf{B} = \mathbf{K}\,\mathbf{B}_U\mathbf{K}^T$ where

$$
\mathbf{K} = \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ K_{ST} & I & 0 & 0 & 0 \\ K_{\eta T} & K_{\eta S} & I & 0 & 0 \\ K_{uT} & K_{uS} & K_{u\eta} & I & 0 \\ K_{vT} & K_{vS} & K_{v\eta} & 0 & I \end{pmatrix}, \quad \mathbf{B}_U = \begin{pmatrix} B_{TT} & 0 & 0 & 0 & 0 \\ 0 & B_{S_U S_U} & 0 & 0 & 0 \\ 0 & 0 & B_{\eta_U \eta_U} & 0 & 0 \\ 0 & 0 & 0 & B_{u_U u_U} & 0 \\ 0 & 0 & 0 & 0 & B_{v_U v_U} \end{pmatrix}
$$

where $B_{TT} \equiv E\left[T'\,T'^T\right] = E\left[T'_B\,T'^T_B\right], \; B_{S_U S_U} \equiv E\left[S'_U\,S'^T_U\right],$ etc.

A strong constraint approach for modelling $\mathbf{B}$

- Consider the special case where $S'_U = \eta'_U = u'_U = v'_U = 0$

$$\mathbf{B} = \underbrace{\begin{pmatrix} I \\ K_{ST} \\ K_{\eta T} \\ K_{uT} \\ K_{vT} \end{pmatrix}}_{\mathbf{K}} B_{TT} \underbrace{\begin{pmatrix} I & K_{ST}^T & K_{\eta T}^T & K_{uT}^T & K_{vT}^T \end{pmatrix}}_{\mathbf{K}^T}$$

- Here $\mathbf{K}$ is a "strong constraint" (*Lorenc 2002*).
- We only need a univariate statistical model for

$$B_{TT} = \Sigma_T C_{TT} \Sigma_T$$

- All other covariances are determined implicitly from B_{TT} using $\mathbf{K}$ and $\mathbf{K}^T$.

A strong constraint approach for modelling **B**

- **B** has a nullspace associated with the “unbalanced” components S'_U, η'_U, u'_U and v'_U .

- The reduced control variable is

$$v_T = \Sigma_T^{-1} \; C_{TT}^{-1/2} \qquad \mathbf{K}^{-I} \qquad \delta\mathbf{x}$$

$$\left(\;\right) = \left(\ddots\right)\left(\quad\right)\left(\qquad\qquad\qquad\right)\begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}$$

A strong constraint approach for modelling **B**

- Recall that for the preconditioned variational problem, we need to specify only the inverse of the change of variable:

$$\delta\mathbf{x} = \mathbf{K}\ \Sigma_T\ C_{TT}^{1/2}\ v_T$$

and its adjoint for computing the gradient of J_o:

$$v_T^* = (C_{TT}^{1/2})^T\ \Sigma_T\ \mathbf{K}^T\ \delta\mathbf{x}^*$$

A strong constraint approach for modelling **B**

- Recall that for the preconditioned variational problem, we need to specify only the inverse of the change of variable:

$$\delta \mathbf{x} = \mathbf{K} \; \Sigma_T \, C_{TT}^{1/2} \, v_T$$

and its adjoint for computing the gradient of J_o :

$$v_T^* = \left(C_{TT}^{1/2}\right)^T \Sigma_T \, \mathbf{K}^T \, \delta \mathbf{x}^*$$

Univariate correlation modelling using a diffusion equation

(Derber & Rosati 1989 - JPO; Egbert et al. 1994 - JGR; Weaver & Courtier 2001 - QJRMS)

1D case:

- Consider $\dfrac{\partial \eta}{\partial t} - \kappa \dfrac{\partial^2 \eta}{\partial z^2} = 0$ with constant $\kappa > 0$.

on $-\infty < z < \infty$ with $\eta(z,t) \rightarrow 0$ as $z \rightarrow \pm\infty$

- Integrate from $t = 0$ and $t = T$ with $\eta(z,0)$ as IC:

$$\eta(z,T) = \frac{1}{\sqrt{4\pi\kappa T}} \int_{z'} e^{-(z-z')^2/4\kappa T} \, \eta(z',0)\, dz'$$

Solution: $\eta(z,T) = \dfrac{1}{\sqrt{4\pi\kappa T}} \displaystyle\int_{z'} e^{-(z-z')^2/4\kappa T} \eta(z',0)\, dz'$

- This integral solution defines, after normalization, a correlation operator $\mathcal{C}$:

$$\eta(z,0) \xrightarrow{\mathcal{C}} \sqrt{4\pi\kappa T}\ \eta(z,T)$$

- The kernel of $\mathcal{C}$ is a Gaussian correlation function

$$f(z;\kappa T) = e^{-z^2/2L^2}$$

where $L = \sqrt{2\kappa T}$ is the length scale.

- *Basic idea* : *To compute the action of* $\mathcal{C}$ *on a discrete grid we can iterate a diffusion operator.*

This is much cheaper than solving an integral equation directly.

Theoretical generalization: a family of isotropic correlation functions on the sphere

(Wahba 1985; Weaver & Courtier 2001-QJRMS)

- Consider the differential operator

$$\eta(\lambda,\phi) = \left(1 - \sum_{p=1}^{P} \alpha_p \, (-\nabla^2)^p \right)^{-M} \hat{\eta}(\lambda,\phi)$$

with constant $\alpha_p > 0$ and integers $M > 0,\ P > 0$.

- Consider solutions of the form

$$\eta(\lambda,\phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \eta_n^m \, Y_n^m(\lambda,\phi)$$

where $Y_n^m(\lambda,\phi)$ are the spherical harmonics, with

$$\nabla^2 Y_n^m = \left(-n\,(n+1)/a^2 \right) Y_n^m$$

Theoretical generalization: a family of isotropic correlation functions on the sphere

- The integral representation of the differential operator is

$$\eta(\lambda,\phi) = \frac{1}{4\pi\, a^2} \int_{\Sigma} f(\theta)\, \hat{\eta}(\lambda,\phi)\, d\Sigma \qquad \text{(A)}$$

where $f(\theta) = \sum_{n=0}^{\infty} f_n\, P_n^0(\cos\theta)$

and $\boxed{f_n(\alpha_p, P, M) = \sqrt{2n+1}\left(1 + \frac{1}{M}\sum_{p=1}^{P} \alpha_p \left(n\,(n+1)/a^2\right)^p\right)^{-M}}$

- The $f_n > 0$ so (A) is a valid (positive definite) covariance operator (*e.g., see Gaspari and Cohn 1999 - QJRMS*).

Theoretical generalization: a family of isotropic correlation functions on the sphere

- The length scale L of the correlation functions can be defined by *(Daley 1991):*

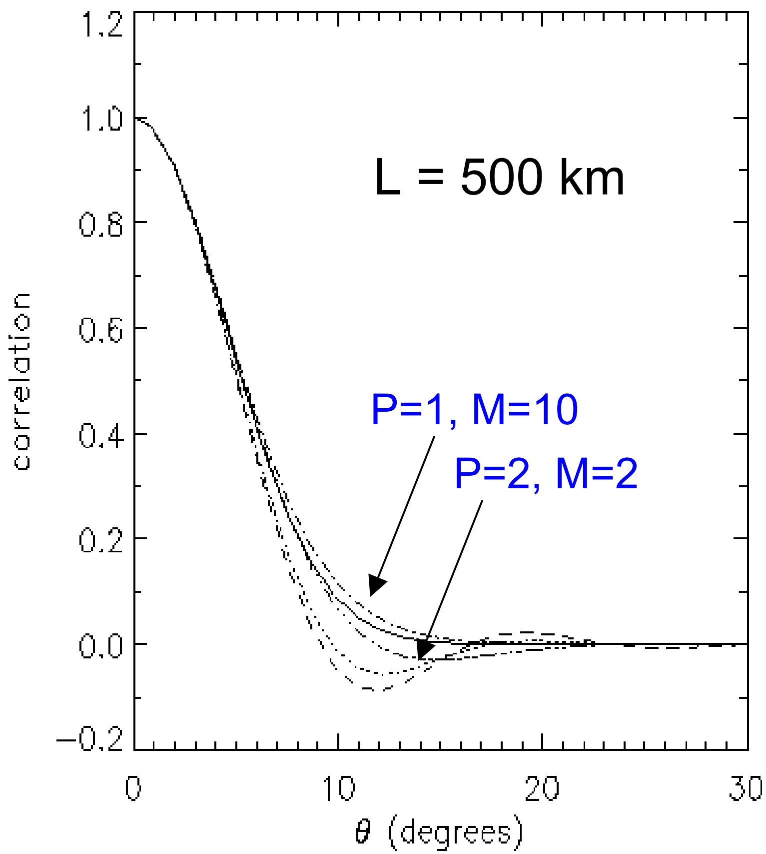
$$L^2 = -\left.\frac{f}{\nabla^2 f}\right|_{\theta=0}$$

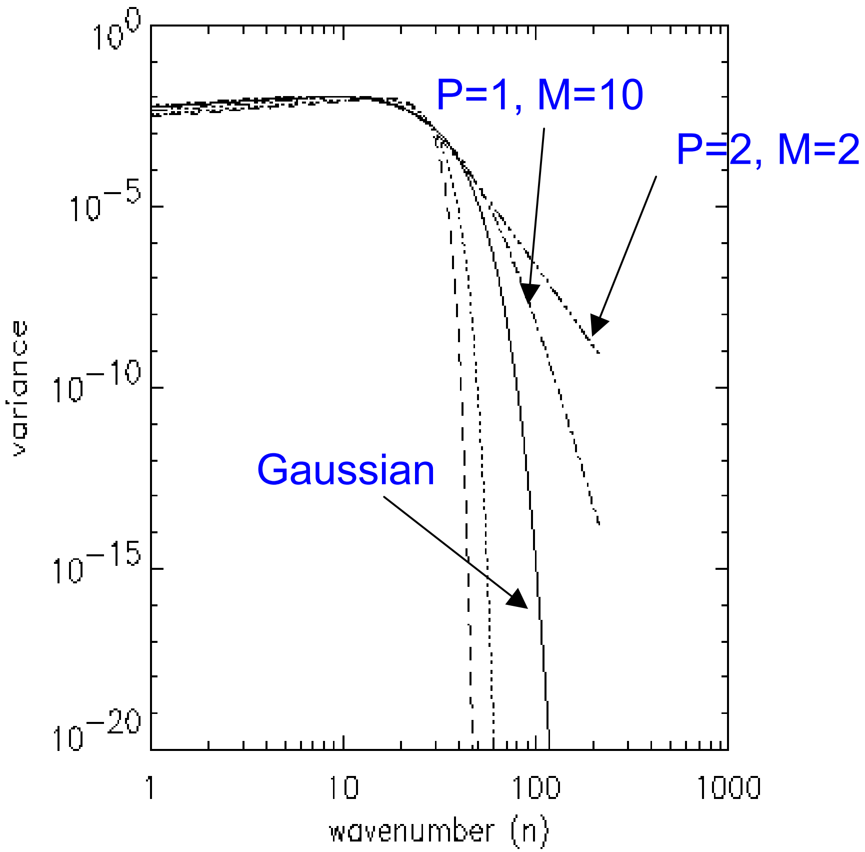
Examples

$$f(\theta) = \sum_{n=0}^{\infty} f_n \, P_n^0(\cos\theta) \qquad f_n = \sqrt{2n+1}\left(1 + \frac{1}{M}\sum_{p=1}^{P} \alpha_p \left(n\,(n+1)/a^2\right)^p\right)^{-M}$$

shape $f(\theta)$

spectrum f_n





Theoretical generalization: a family of isotropic correlation functions on the sphere

- We can identify the previous differential operator as the solution of a generalized diffusion equation (GDE)

$$\boxed{\frac{\partial \eta}{\partial t} + \sum_{p=1}^{P} \kappa_p \left(-\nabla^2\right)^p \eta = 0}$$

using implicit time discretization

$$\eta(\lambda,\phi,T) = \left(1 - \sum_{p=1}^{P} \kappa_p \Delta t \, (-\nabla^2)^p \right)^{-M} \eta(\lambda,\phi,0) \qquad \text{(A)}$$

where

$$\kappa_p \, \Delta t \leftrightarrow \alpha_p \quad ; \quad \eta(\lambda,\phi,0) \leftrightarrow \hat{\eta}(\lambda,\phi)$$

$$T \leftrightarrow M \, \Delta t \quad ; \quad \eta(\lambda,\phi,T) \leftrightarrow \eta(\lambda,\phi)$$

- We can use direct or iterative algorithms for solving (A) in grid-point space.

Some remarks on numerical implementation

- The full correlation operator is formulated in grid-point space as a sequence of operators

$$\mathbf{C} = \mathbf{\Lambda}\, \mathbf{L}^{1/2}\, \mathbf{W}^{-1}\, \mathbf{L}^{T/2}\, \mathbf{\Lambda}$$
$$= \underbrace{\left(\mathbf{\Lambda}\, \mathbf{L}^{1/2}\, \mathbf{W}^{-1/2}\right)}_{\mathbf{C}^{1/2}} \underbrace{\left(\mathbf{W}^{-1/2}\, \mathbf{L}^{T/2}\, \mathbf{\Lambda}\right)}_{\mathbf{C}^{T/2}}$$

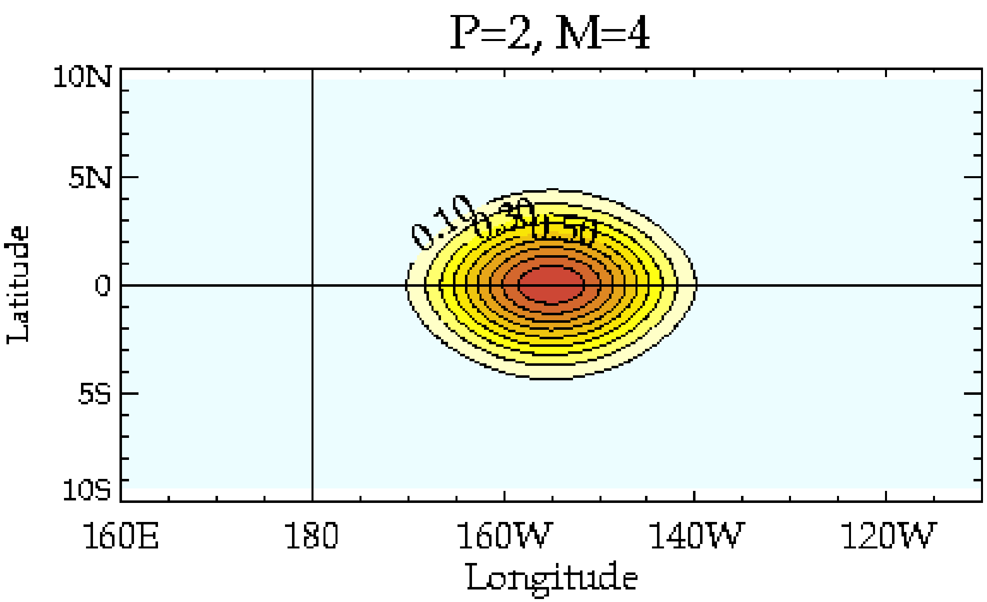
- $\mathbf{L}$ is the diffusion operator and is formulated in 3D as a product $\mathbf{L} = \mathbf{L}_h\, \mathbf{L}_v$ of a 2D (horizontal) and 1D (vertical) operator.

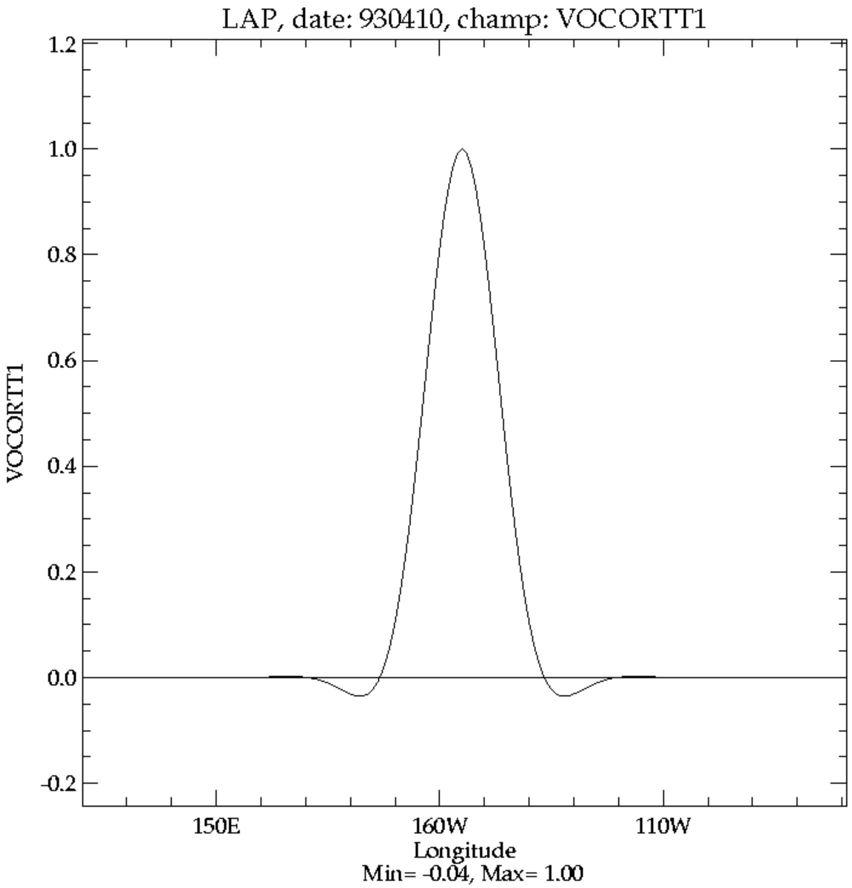
- $\mathbf{W}$ is a diagonal matrix of volume elements, and appears in $\mathbf{C}$ because of the self-adjointness of $\mathbf{L}$.

- The factor $\mathbf{L}^{1/2}$ means $M/2$ iterations of the diffusion operator.

GDE-generated correlation functions

Example: T-T correlations at the equator





Some remarks on numerical implementation

- We can let $\nabla^2 \rightarrow \nabla \cdot R \nabla$ where R is a diffusion tensor that can be used to stretch and/or rotate the coordinates in the correlation model to account for anisotropic or flow-dependent structures.

- BCs are imposed directly within the discrete expression for ∇^2 using a land-ocean mask.

- $\mathbf{\Lambda}$ contains normalization factors to ensure the variances of $\mathbf{C}$ are equal to one.

- The diffusion and recursive filter *(Lorenc 1991 – QJRMS; Purser et al. 2003 - MWR; Derber lecture)* approaches to correlation modelling have many similarities.

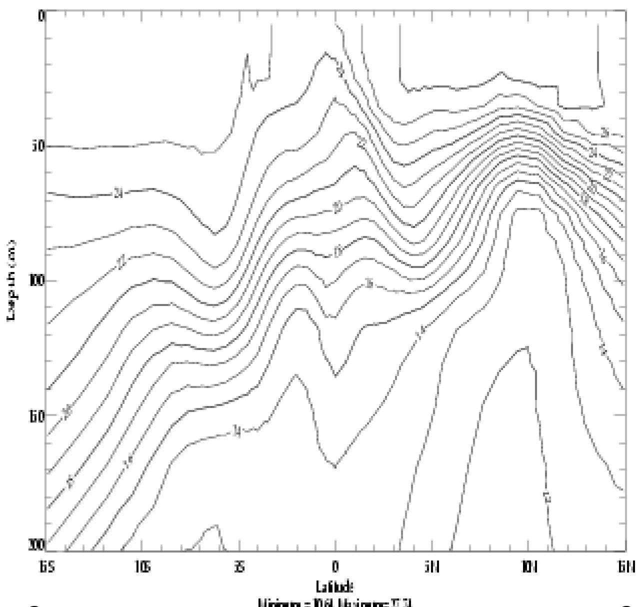
GDE-generated correlation functions

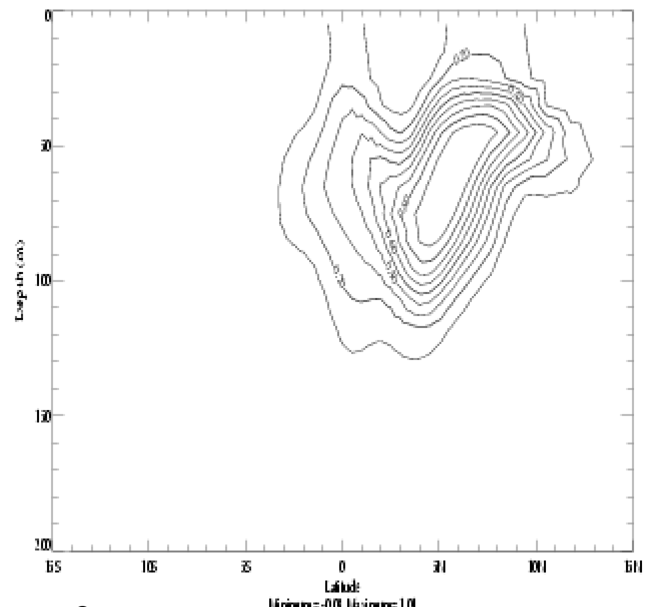
Example: flow-dependent correlations

(Weaver & Courtier 2001-QJRMS; cf. Riishojgaard 1998-Tellus; Daley & Barker 2001-MWR)

Background isothermals | T-T correlations

Depth





15^{o}S 15^{o}N 15^{o}S 15^{o}N

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The normalization matrix $\mathbf{\Lambda}$

- The elements of $\mathbf{\Lambda}$ are the inverse of the square root of the diagonal elements of the cov. filter

$$\mathbf{L}^{1/2}\,\mathbf{W}^{-1}\mathbf{L}^{T/2}$$

- With constant "diffusion" coefficient and in the absence of boundaries

$$\mathbf{\Lambda} = \lambda\,\mathbf{I}$$

- With spatially varying "diffusion" coefficients or in the presence of boundaries

$$\mathbf{\Lambda} = \mathrm{diag}\left(\lambda_i\right), \quad i = 1,\ldots,\ N$$

Computing the normalization factors

- Define the square root of the covariance filter

$$\hat{\mathbf{v}} = \mathbf{L}^{1/2}\,\mathbf{W}^{-1/2}\mathbf{v}$$

- Algorithm 1 : Exact
  - Let $\mathbf{v} = (0,\ldots,0,\underset{\uparrow}{1},0,\ldots,0)$

    i-th grid point
  - Then $\lambda_i^{-2} = \hat{\mathbf{v}}^T\,\hat{\mathbf{v}}$
  - One application of the square root filter is needed for each grid point *i* $\Rightarrow$ Expensive!

Computing the normalization factors

- Define the square root of the covariance filter

$$\hat{\mathbf{v}} = \mathbf{L}^{1/2}\,\mathbf{W}^{-1/2}\mathbf{v}$$

- Algorithm 2 : Randomization
  (Fisher & Courtier 1995; Andersson lecture)

  – Choose $\mathbf{v}$ such that $E\left[\mathbf{v}\right] = 0$ and $E\left[\mathbf{v}\,\mathbf{v}^T\right] = \mathbf{I}$

  – Then $\lambda_i^{-2} \approx \operatorname{diag}_i\left(\frac{1}{Q-1}\sum_{q=1}^{Q}\hat{\mathbf{v}}_q\,\hat{\mathbf{v}}_q^T\right)$

  – The estimate of λ_i improves as Q gets large.

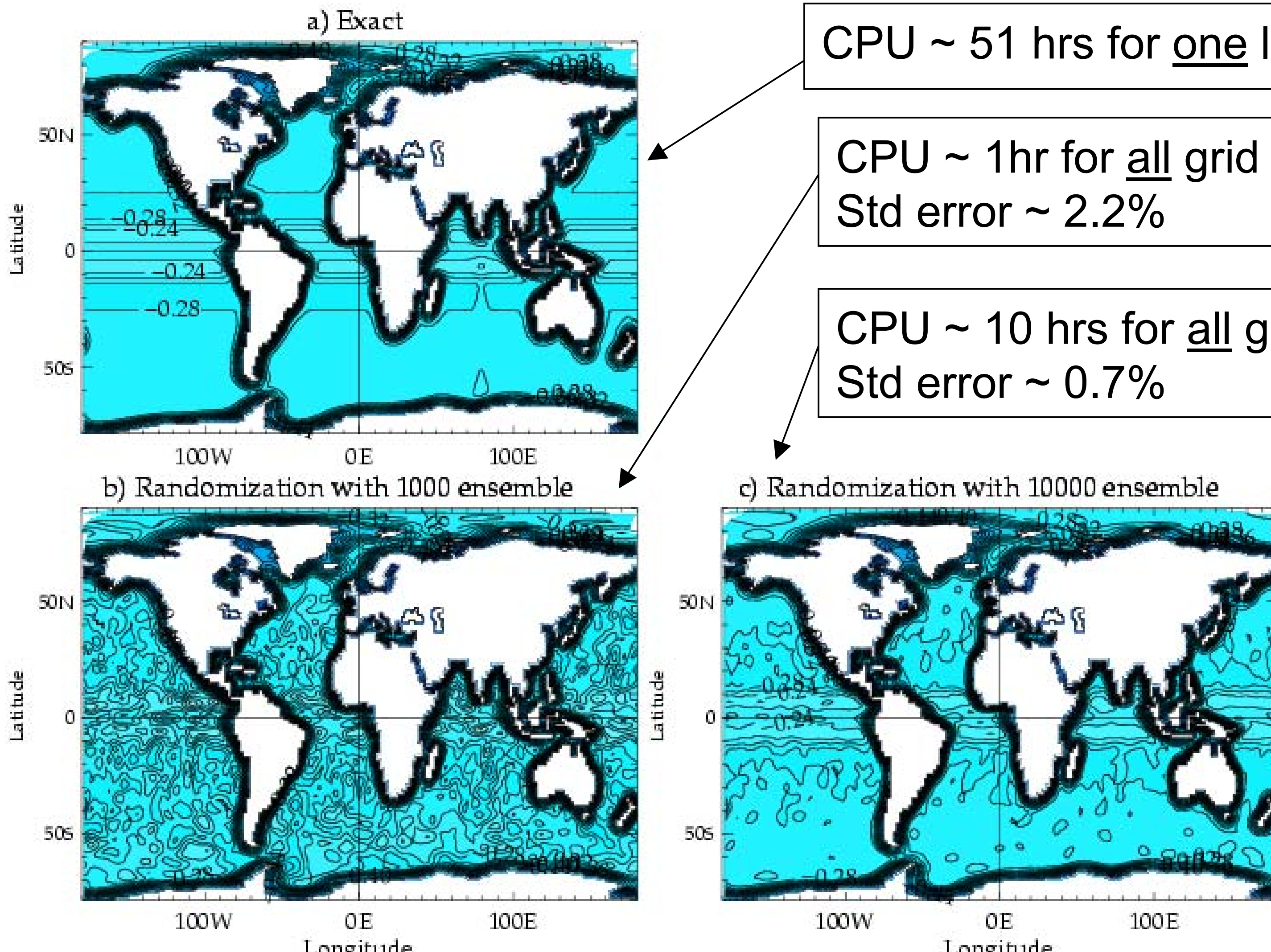
  – Estimate of the randomization error = $1/\sqrt{2Q}$

Computing the normalization factors

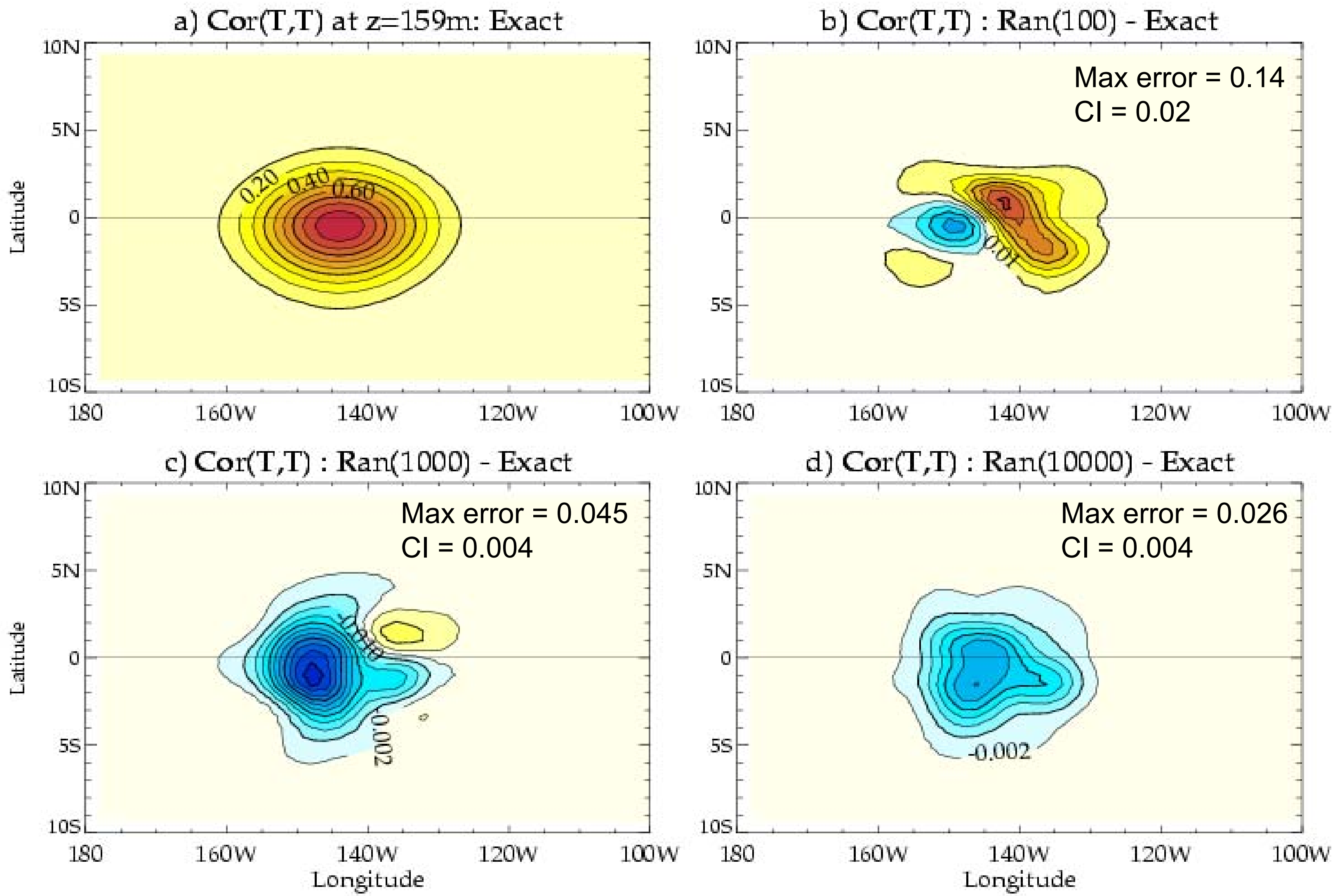
CPU ~ 51 hrs for <u>one</u> level!

CPU ~ 1hr for <u>all</u> grid pts
Std error ~ 2.2%

CPU ~ 10 hrs for <u>all</u> grid pts
Std error ~ 0.7%



Impact of randomization on the correlations



A strong constraint approach for modelling **B**

- Recall that for the preconditioned variational problem we need to specify only the inverse of the change of variable

$$\delta\mathbf{x} = \mathbf{K}\,\Sigma_T\, C_{TT}^{1/2}\, v_T$$

and its adjoint for computing the gradient of J_o:

$$v_T^* = (C_{TT}^{1/2})^T\,\Sigma_T\,\mathbf{K}^T\,\delta\mathbf{x}^*$$

A flow-dependent parametrisation for Σ_T

(Behringer et al. 1998 - MWR ; Alves et al. 2003 - QJRMS)

- Assume that the elements σ_T^b of Σ_T are a function of the background vertical temperature gradient $\partial T^b / \partial z$.

Physical justification:

- Errors in the temperature (T) state are expected to be largest in regions of strong variability; e.g., in the thermocline where $\partial T^b / \partial z$ is large.
- Assuming that the background T profile (T^b) and "true" T profile (T^t) differ because of a vertical displacement error δz then (*cf. Cooper and Haines 1996 – JGR*)

$$T^t(z) = T^b(z + \delta z) \approx T^b(z) + \underbrace{\left(\partial T^b / \partial z\right) \times \delta z}_{\varepsilon^b}$$

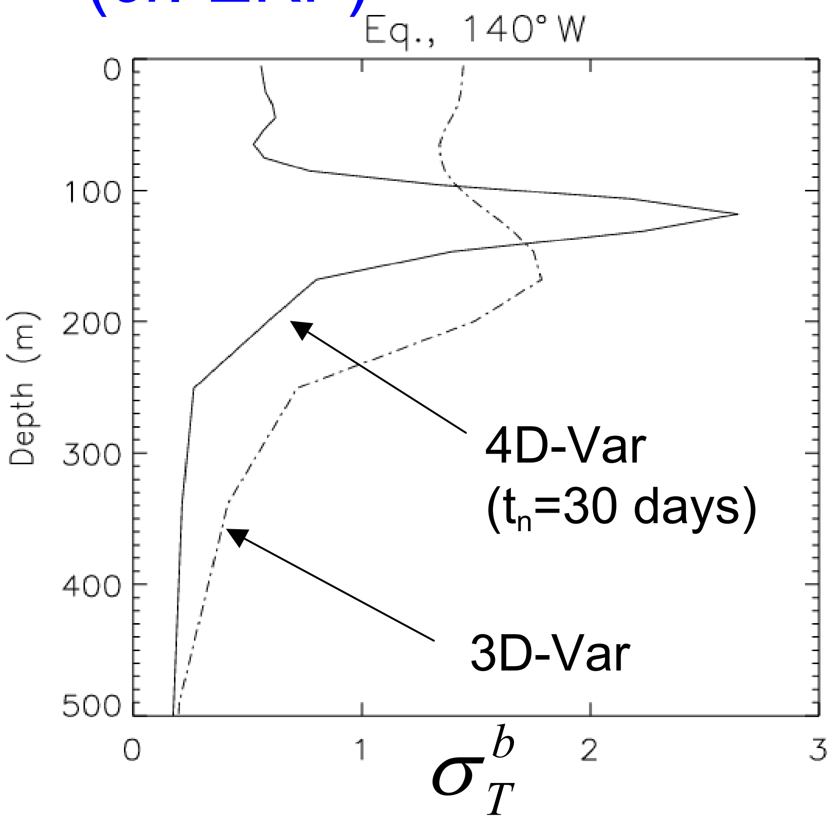
Diagnosing σ_T^b in 4D-Var

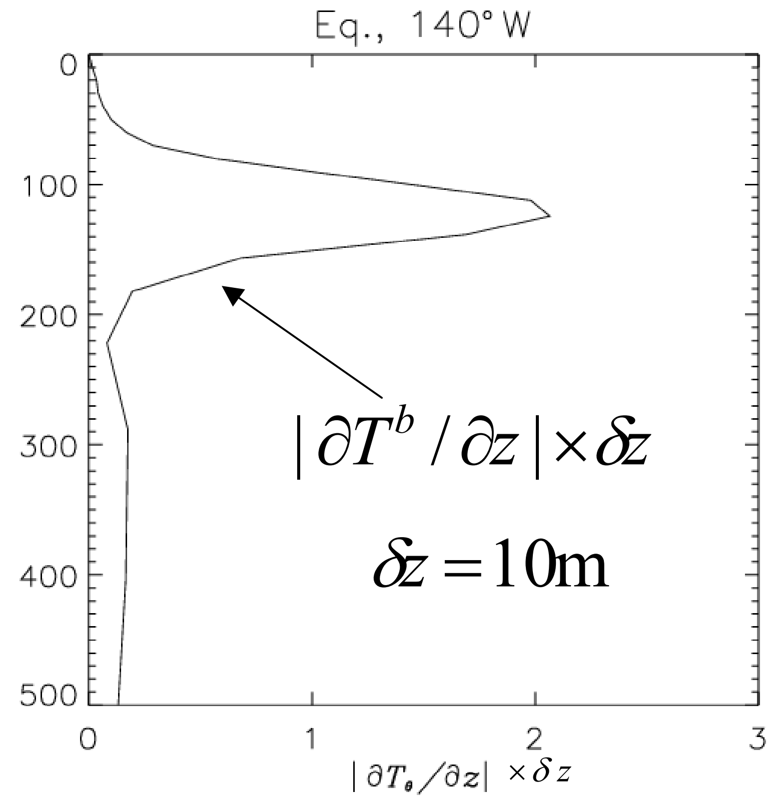
(Weaver et al. 2003 – MWR)

In 3D-Var: $\mathbf{P}^b(t_n) = \mathbf{B}$

In 4D-Var: $\mathbf{P}^b(t_n) = \mathbf{M}(t_n, t_0)\,\mathbf{B}\,\mathbf{M}(t_n, t_0)^T$

(*cf.* EKF)





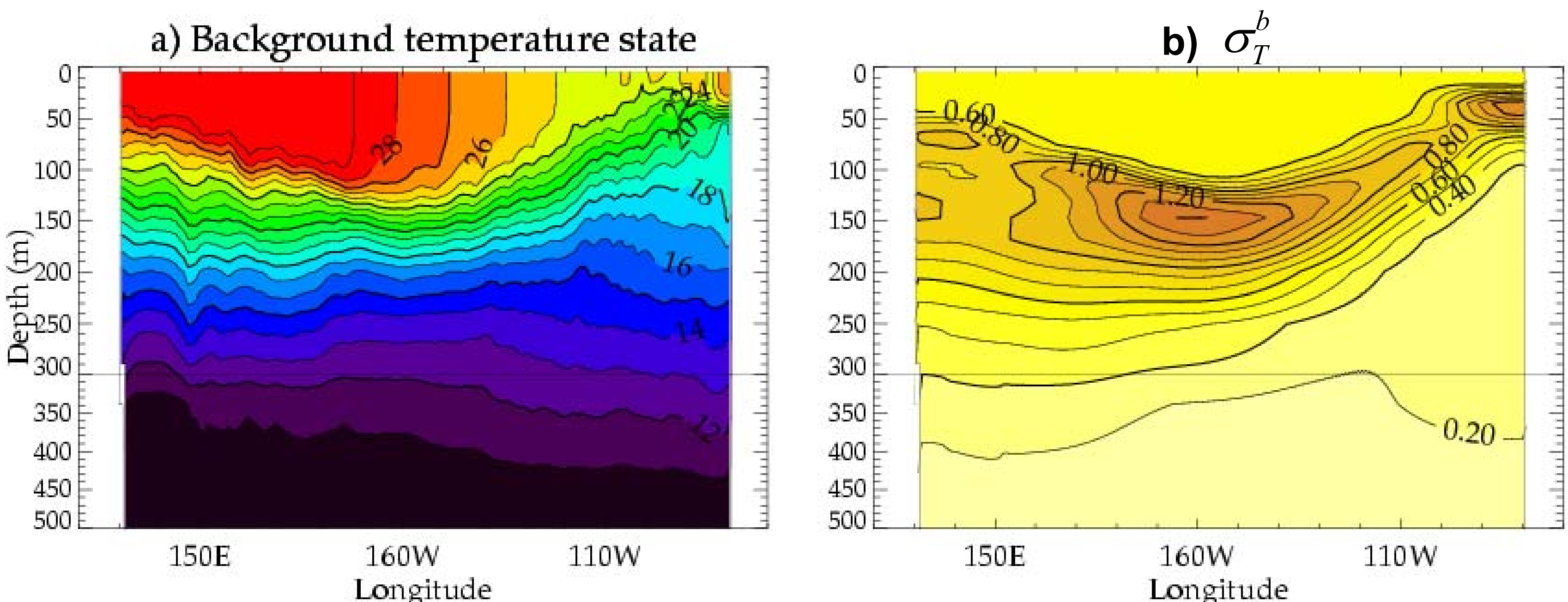
A flow-dependent parametrisation for σ_T^b

Example:

$$\sigma_T^b(\lambda,\phi,z) = \begin{cases} \sigma_{\min}^b & \text{in the mixed layer} \\ \min\left(\left(\partial T^b / \partial z\right)\times \delta z\, ,\, \sigma_{\max}^b\right) & \text{below the mixed layer} \end{cases}$$

with $\sigma_{\min}^b = 0.5^\circ\mathrm{C}$, $\sigma_{\max}^b = 1.5^\circ\mathrm{C}$ and $\delta z = 10\mathrm{m}$

$\sigma_T^b(\lambda,\phi,z)$ is smoothed in each level using the correlation filter



A strong constraint approach for modelling **B**

- Recall that for the preconditioned variational problem, we need to specify only the inverse of the change of variable:

$$\delta\mathbf{x} = \mathbf{K}\,\Sigma_T\, C_{TT}^{1/2}\, v_T$$

and its adjoint for computing the gradient of J_o:

$$v_T^* = (C_{TT}^{1/2})^T\,\Sigma_T\,\mathbf{K}^T\,\delta\mathbf{x}^*$$

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(cf. Derber & Bouttier 1999 - Tellus)

- Suppose (to be justified shortly)

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\end{aligned}
$$

A flow-dependent model for K_{ST}

(Ricci et al. 2003)

- Use a local T-S relation from the background state

  (Troccoli and Haines 1999 - JAOT; Troccoli et al. 2002 - JPO)

$$S^b = S(T^b)$$

- Water mass T-S properties are largely preserved in regions where isentropic processes dominate (e.g., in the tropical thermocline).

- For small perturbations T' about T^b

$$S(T) = S(T^b + T') \cong S(T^b) + \underbrace{\left(\frac{\partial S}{\partial T}\right)\Bigg|_{\substack{S=S^b \\ T=T^b}} \times T'}_{S'}$$

A flow-dependent model for K_{ST} cont.

- Assume that local perturbations to the T-S relation arise through vertical displacements of the background isopycnals :

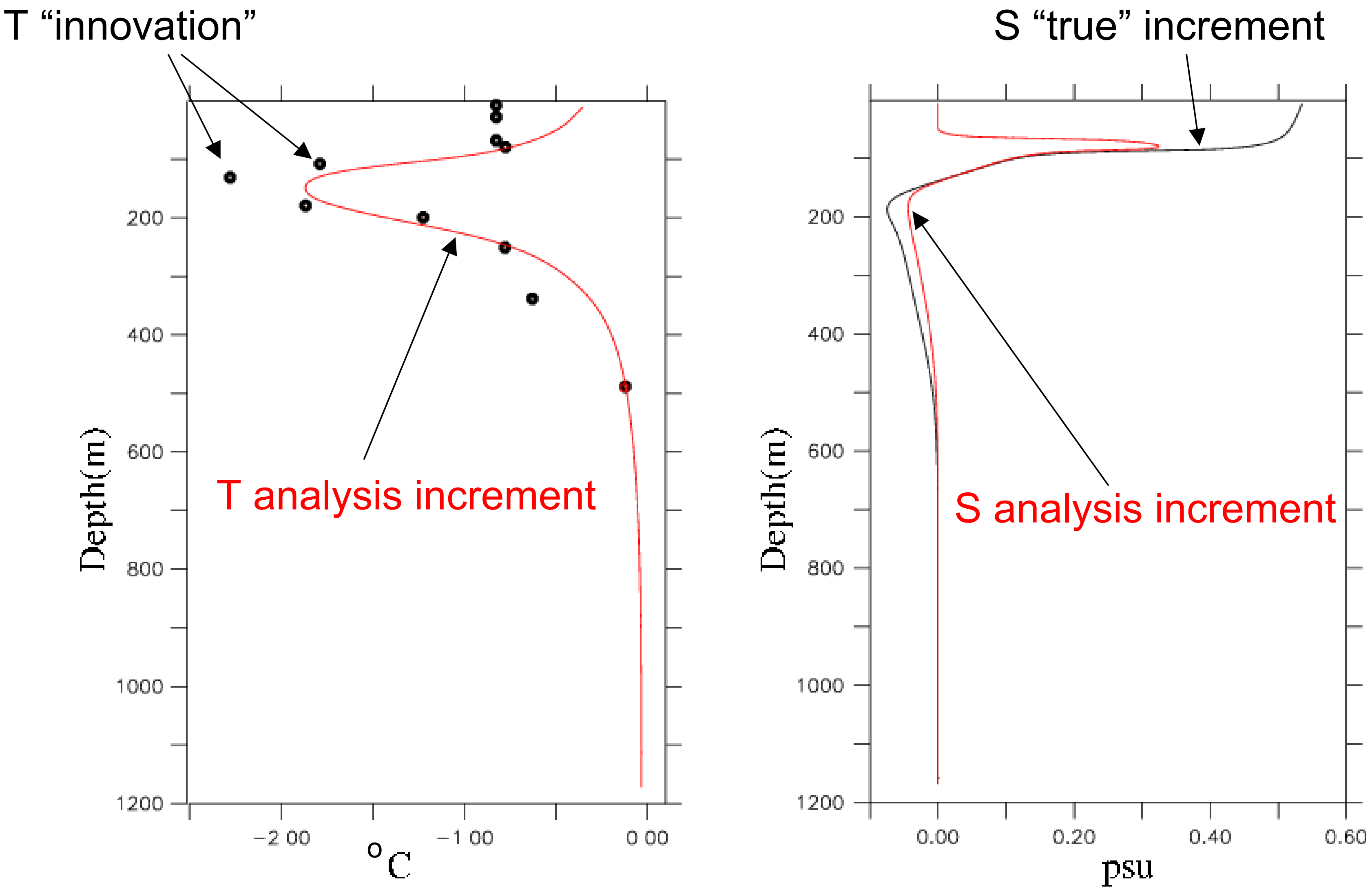
$$\partial S / \partial T\Big|_{S=S^b, T=T^b} = \frac{\partial S / \partial z\big|_{S=S^b}}{\partial T / \partial z\big|_{T=T^b}}$$

- We avoid applying a T-S constraint in regions where nonisentropic processes are important (e.g., in the mixed layer) :

$$S' = w(\mathbf{x}^b) \times \partial S / \partial T\Big|_{S=S^b, T=T^b} \times T'$$

where $w(\mathbf{x}^b) = 0$ or 1 depending on conditions in $\mathbf{x}^b$.

Impact of the multivariate T-S constraint in 3D-Var: a twin experiment.



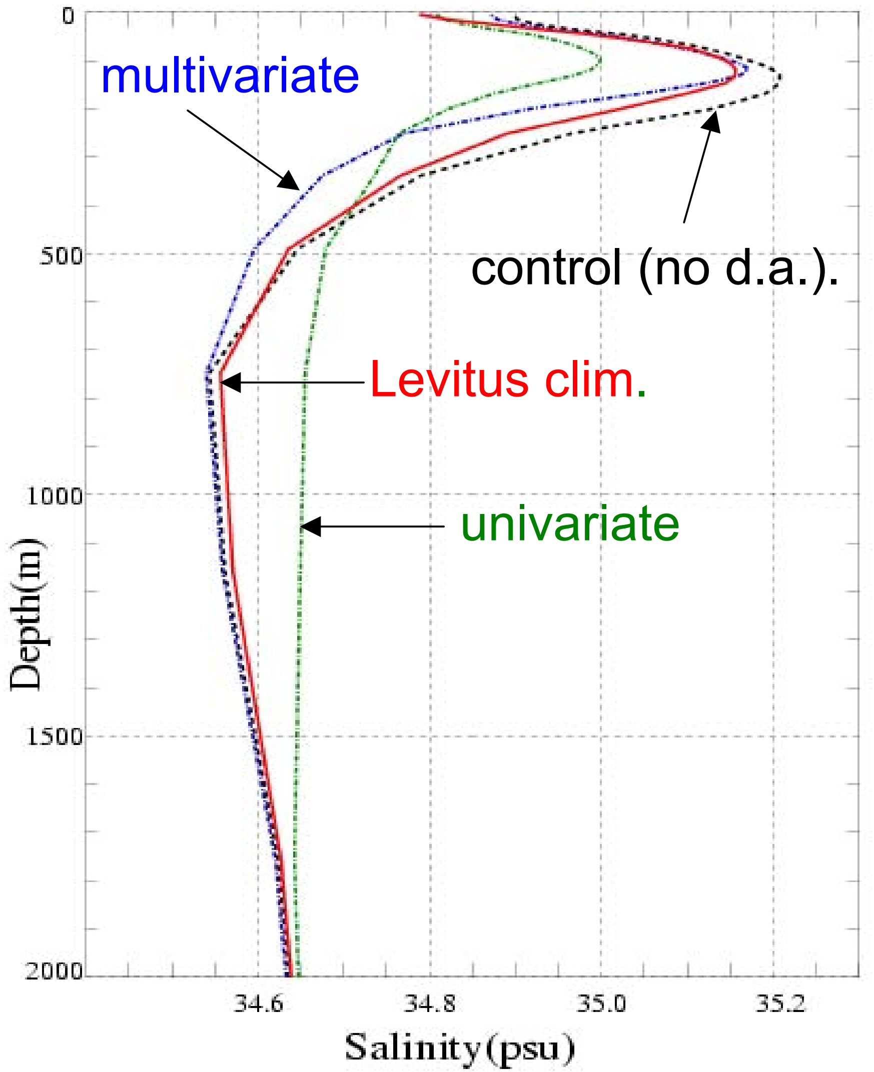
Impact of the multivariate T-S constraint on the salinity mean state in 3D-Var

- <u>Univariate (T) case</u>:

- Spurious circulation develops.
- Artificial decrease / increase of the salinity in the upper / deeper ocean.
- Destruction of the salinity maximum.

- <u>Multivariate (T-S) case</u>:

- Realistic dynamical balances restored.
- Better conservation of water masses.
- Salinity maximum restored.

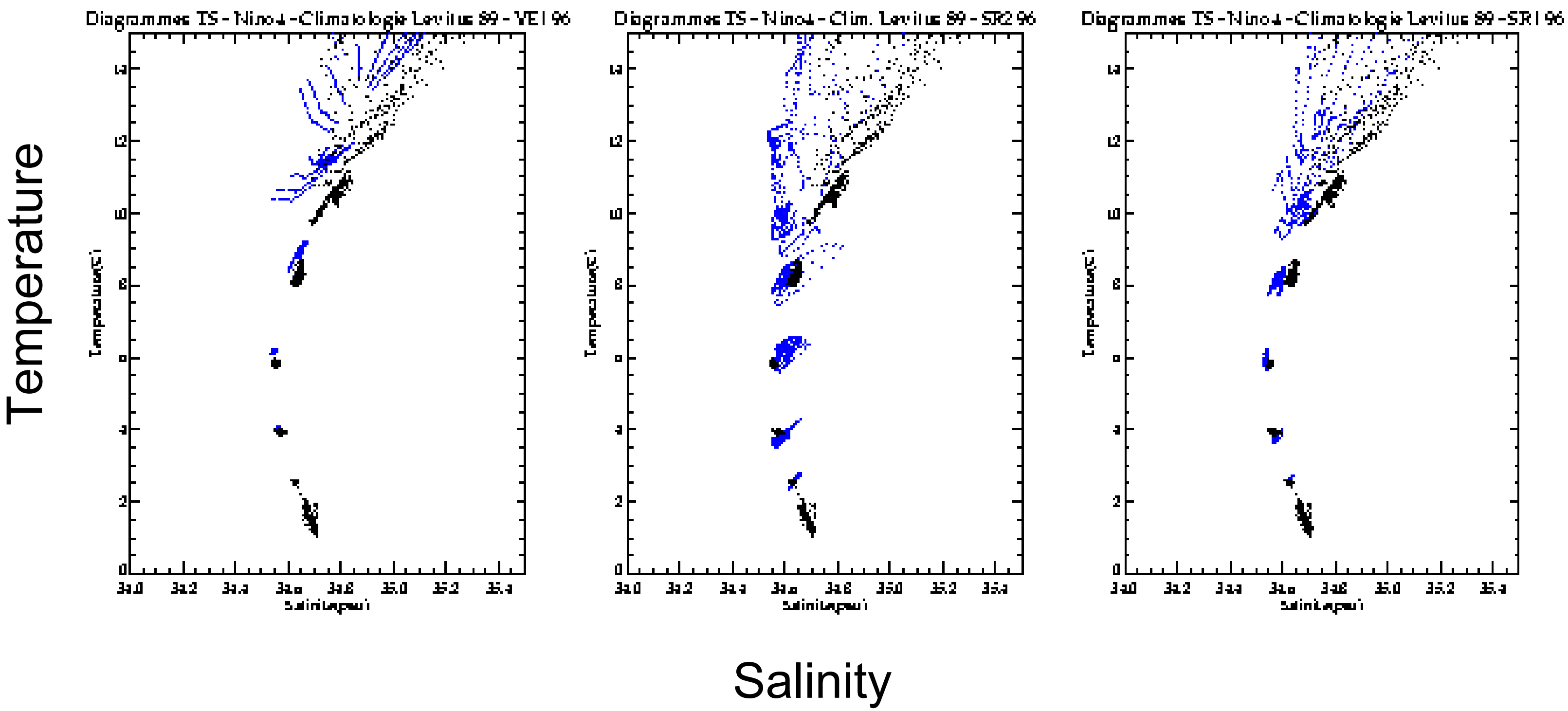


Multivariate T-S constraint

Effect on water masses in 3D-Var

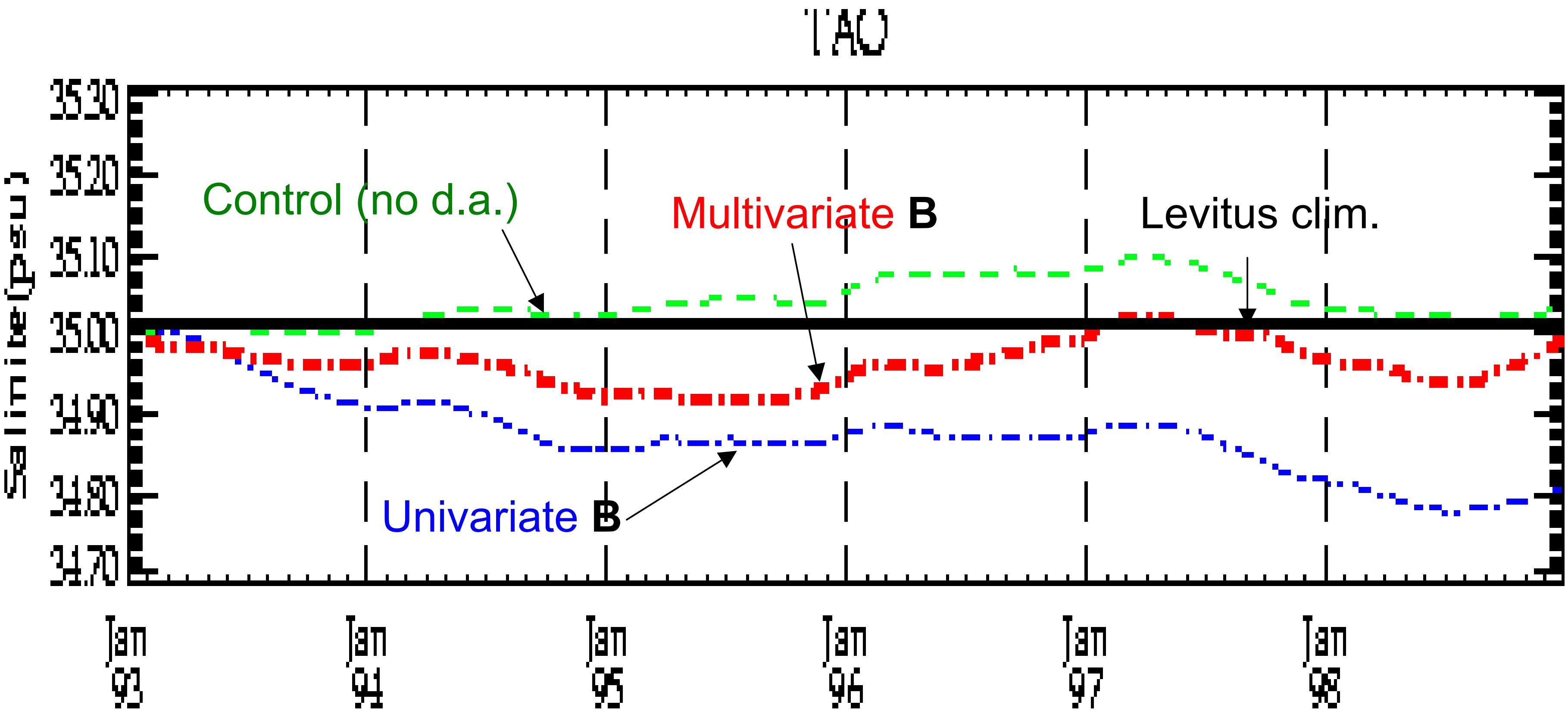
Control (no d.a.) | Univariate **B** | Multivariate **B**

Levitus | Levitus | Levitus



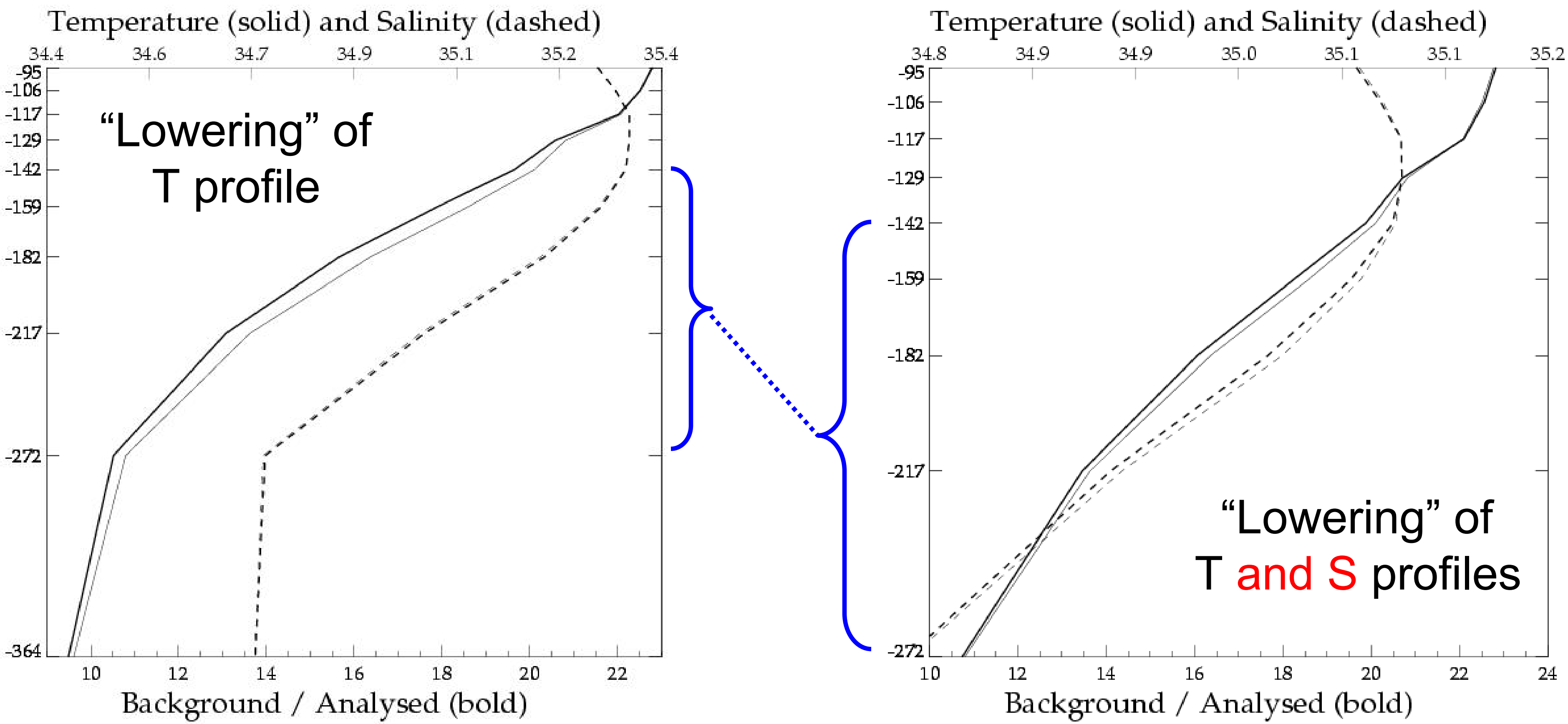
Multivariate T-S constraint

Effect on salinity drift in 3D-Var



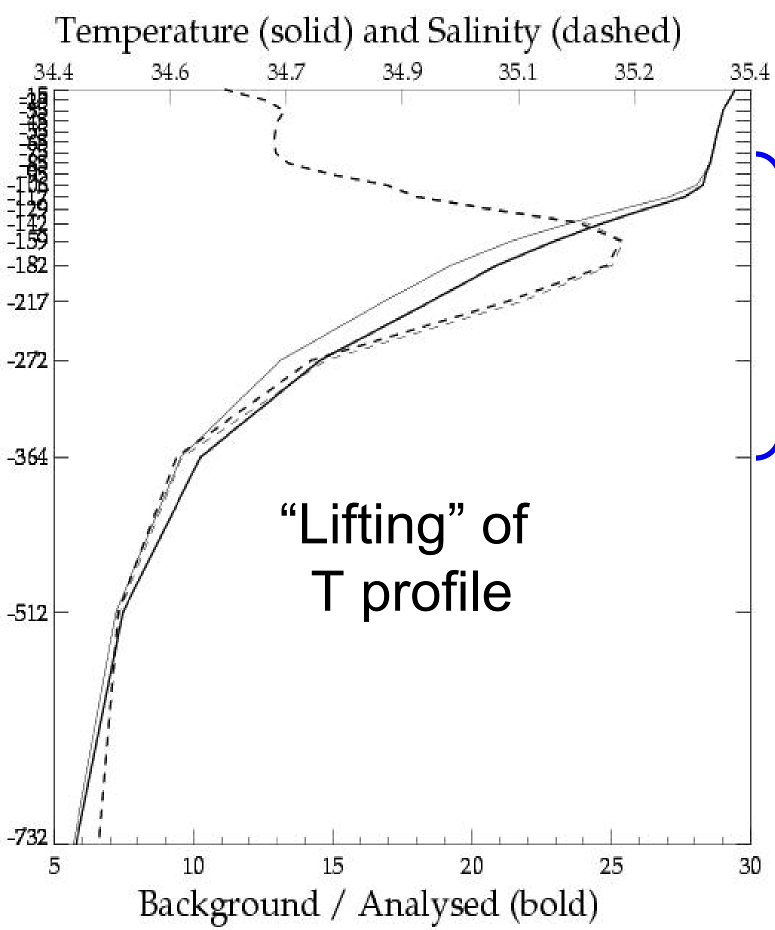
Impact of the multivariate T-S constraint in 4D-Var: a single SSH obs experiment

Univariate **B** — Multivariate **B**

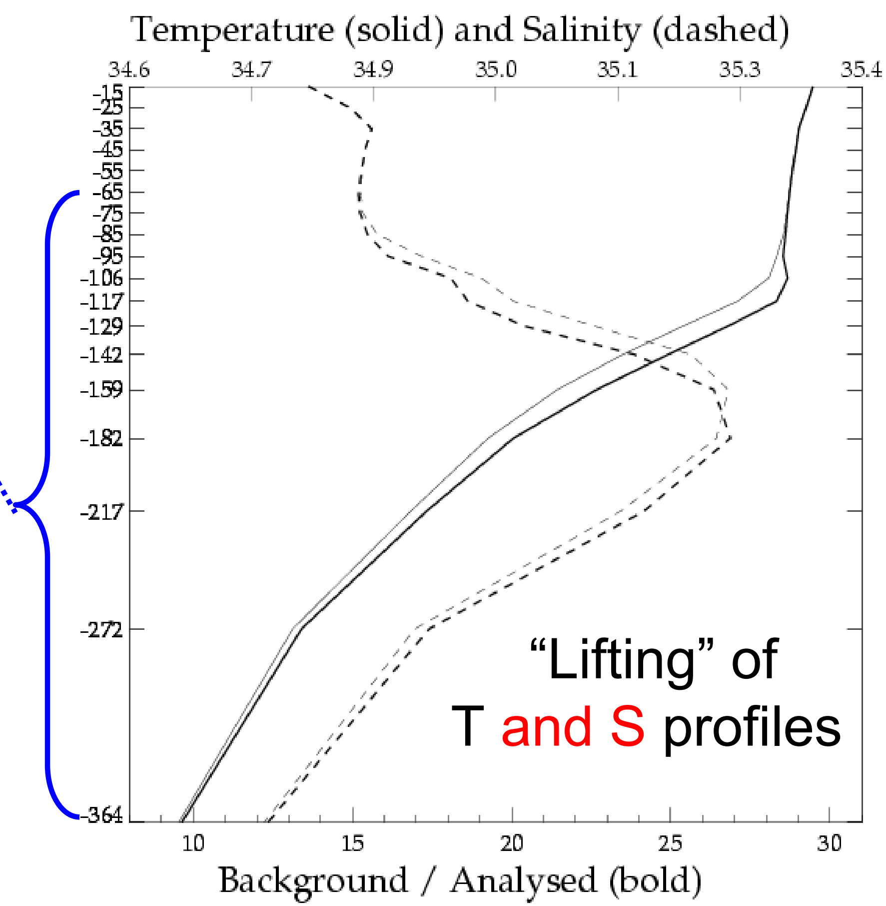


Impact of the multivariate T-S constraint in 4D-Var: a single SSH obs experiment

Univariate **B**



Multivariate **B**



A general approach for modelling **B**

(cf. Derber & Bouttier 1999 - Tellus)

- Suppose (to be justified shortly)

"balanced" variables

"unbalanced" variables

$$
\begin{aligned}
T' &= T'_B \\
S' &= S'_B + S'_U = \underbrace{K_{ST}T'}_{S'_B} + S'_U \\
\eta' &= \eta'_B + \eta'_U = \underbrace{K_{\eta T}T' + K_{\eta S}S'}_{\eta'_B} + \eta'_U \\
u' &= u'_B + u'_U = \underbrace{K_{uT}T' + K_{uS}S' + K_{u\eta}\eta'}_{u'_B} + u'_U \\
v' &= v'_B + v'_U = \underbrace{K_{vT}T' + K_{vS}S' + K_{v\eta}\eta'}_{v'_B} + v'_U
\end{aligned}
$$

A flow-dependent model for $K_{\eta T}$ and $K_{\eta S}$

- Linearized equation of state :

$$\rho' / \rho_0 = -\alpha T' + \beta S'$$

where $\alpha = \partial\rho / \partial T\big|_{T=T^b}$ and $\beta = \partial\rho / \partial S\big|_{S=S^b}$.

- Dynamic height of the surface relative to $z = z_{ref}$:

$$\eta' = -\int_{z_{ref}}^{0} (\rho' / \rho_0)\, dz$$

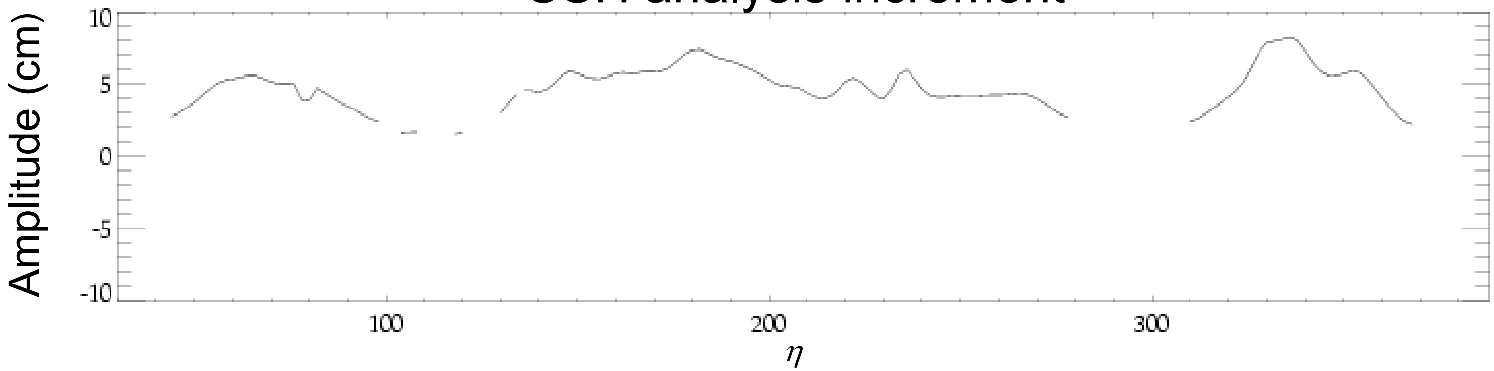
where ρ_0 is a reference density.

Impact of a single SSH obs in 4D-Var

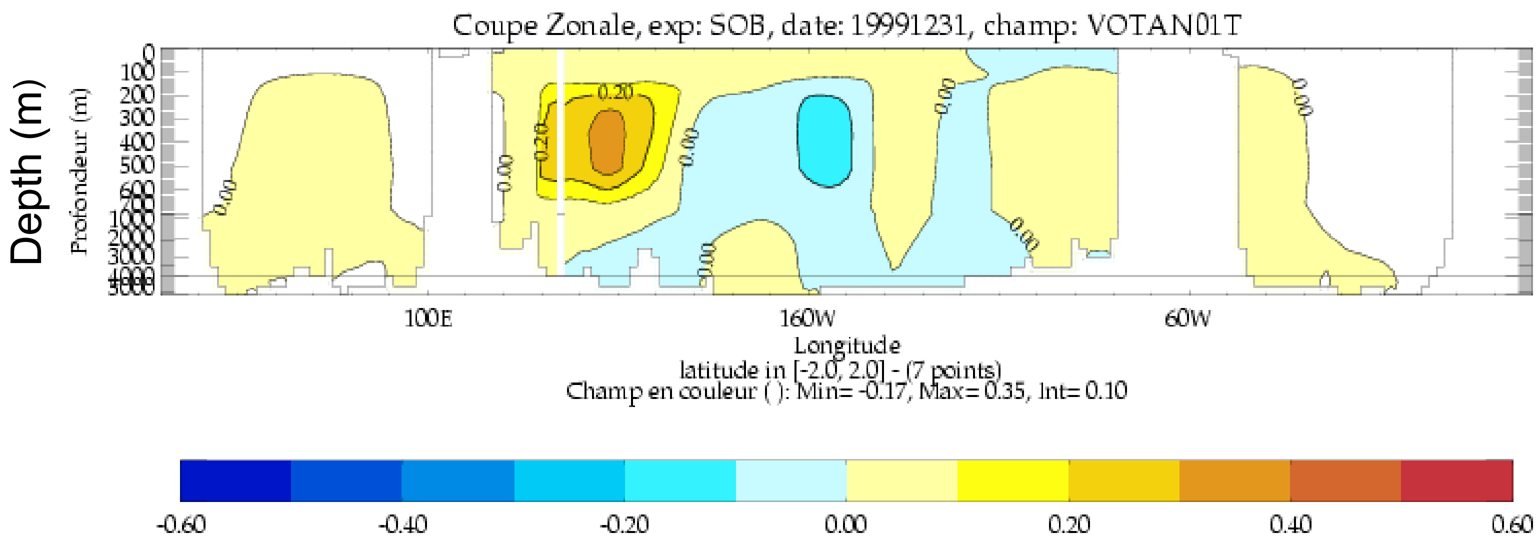
Ex: SSH innovation = 10 cm at (0^o,160^{o}W) at t = 30 days.

a) univariate **B** ; b) constant σ_T^b in the upper 600m.

SSH analysis increment

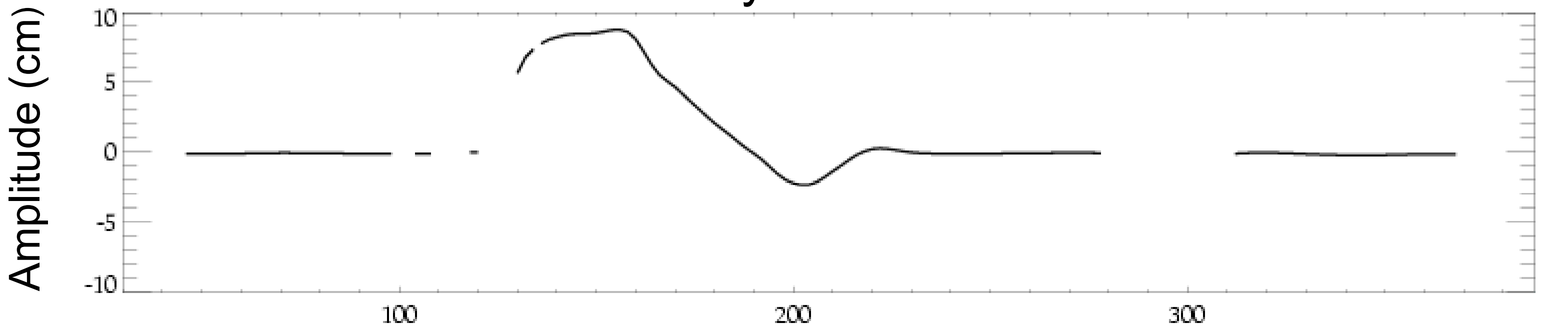


Temperature analysis increment

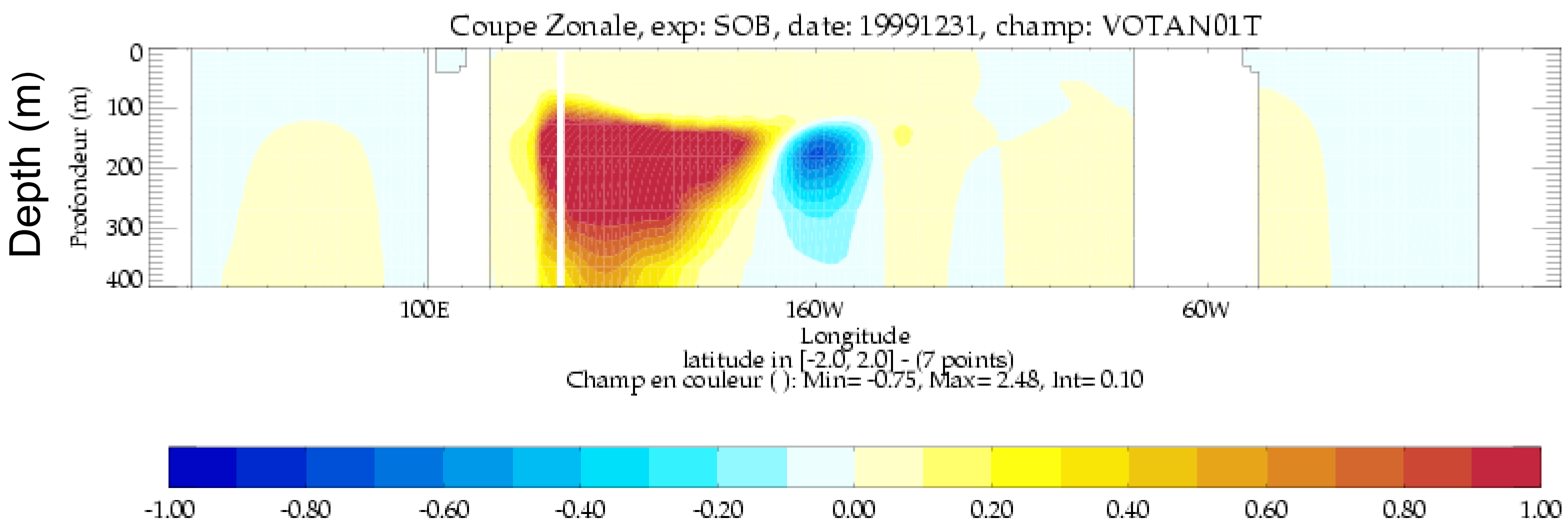


Ex: As previous example but with: a) a multivariate **B** ; b) an extra constraint in **B** to enforce $\overline{\eta'} = 0$; and c) vertical *T*-gradient dependent σ_T^b

SSH analysis increment



Temperature analysis increment



A model for $K_{u(T,S,\eta)}, K_{v(T,S,\eta)}$

- Hydrostatic approximation :

$$p' = \int_{z}^{0} \rho' \, g \, dz + \rho_0 \, g \, \eta'$$

- Combine hydrostatic and dynamic height relations :

$$p' = \int_{z_{ref}}^{z} \rho' \, g \, dz$$

- Geostrophic (f-plane) approximation :

$$f \, u' = -\,(1/\rho_0)(\partial p'/\partial y)$$

$$f \, v' = \;\; (1/\rho_0)\,(\partial p'/\partial x)$$

A model for $K_{u(T,S,\eta)}, K_{v(T,S,\eta)}$ near the equator

(cf. Burgers et al. 2002 - JPO ; Balmaseda lecture)

- Geostrophic f-plane approximation :

$$f\, u'_f = -\,(1/\rho_0)(\partial p'/\partial y)$$

$$f\, v'_f = \;\; (1/\rho_0)\,(\partial p'/\partial x)$$

- Geostrophic β-plane ($f = \beta\, y$) approximation :

$$\beta\, u'_\beta = -\,(1/\rho_0)(\partial^2 p'/\partial y^2)$$

$$\beta\, v'_\beta = \;\; (1/\rho_0)\,(\partial^2 p'/\partial x\,\partial y)$$

A model for $K_{u(T,S,\eta)}, K_{v(T,S,\eta)}$ near the equator

- Combining the f-plane and β-plane solutions (*Lagerloef et al. 1999 – JGR)* :

$$u' = W_\beta \, u'_\beta + (1 - W_\beta) \, u'_f$$

$$v' = W_\beta \, v'_\beta + (1 - W_\beta) \, v'_f$$

where $W_\beta = \exp(-y^2 / 2L_\beta^2).$

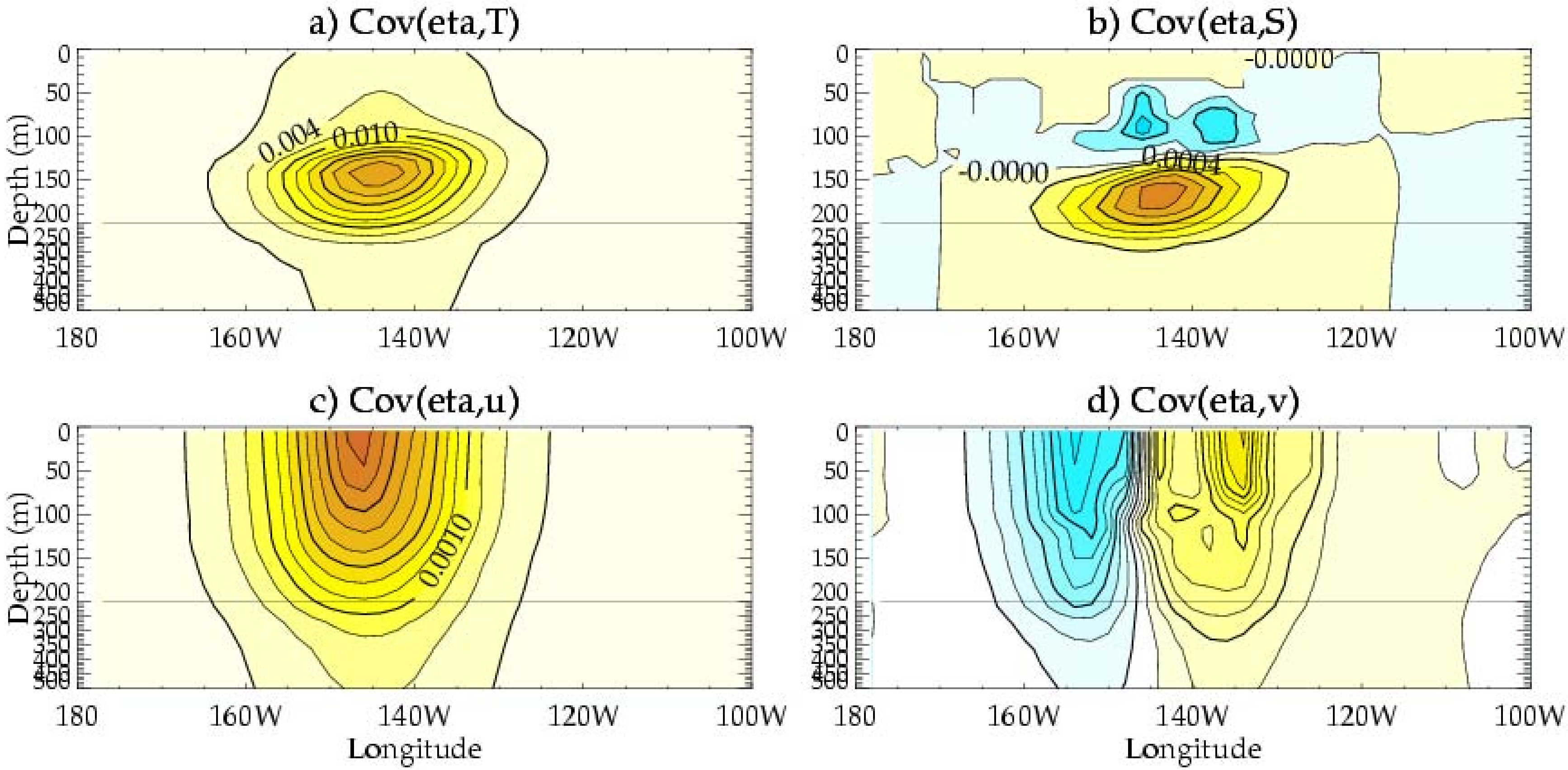
- L_β is a length scale ~ O(eq. Rossby radius) ~ 1° - 2°.

- More elaborate linear models (e.g., which include surface forcing, friction) could be used.

(e.g., Lagerloef et al. 1999-JGR; Bonjean & Lagerloef 2002-JPO)

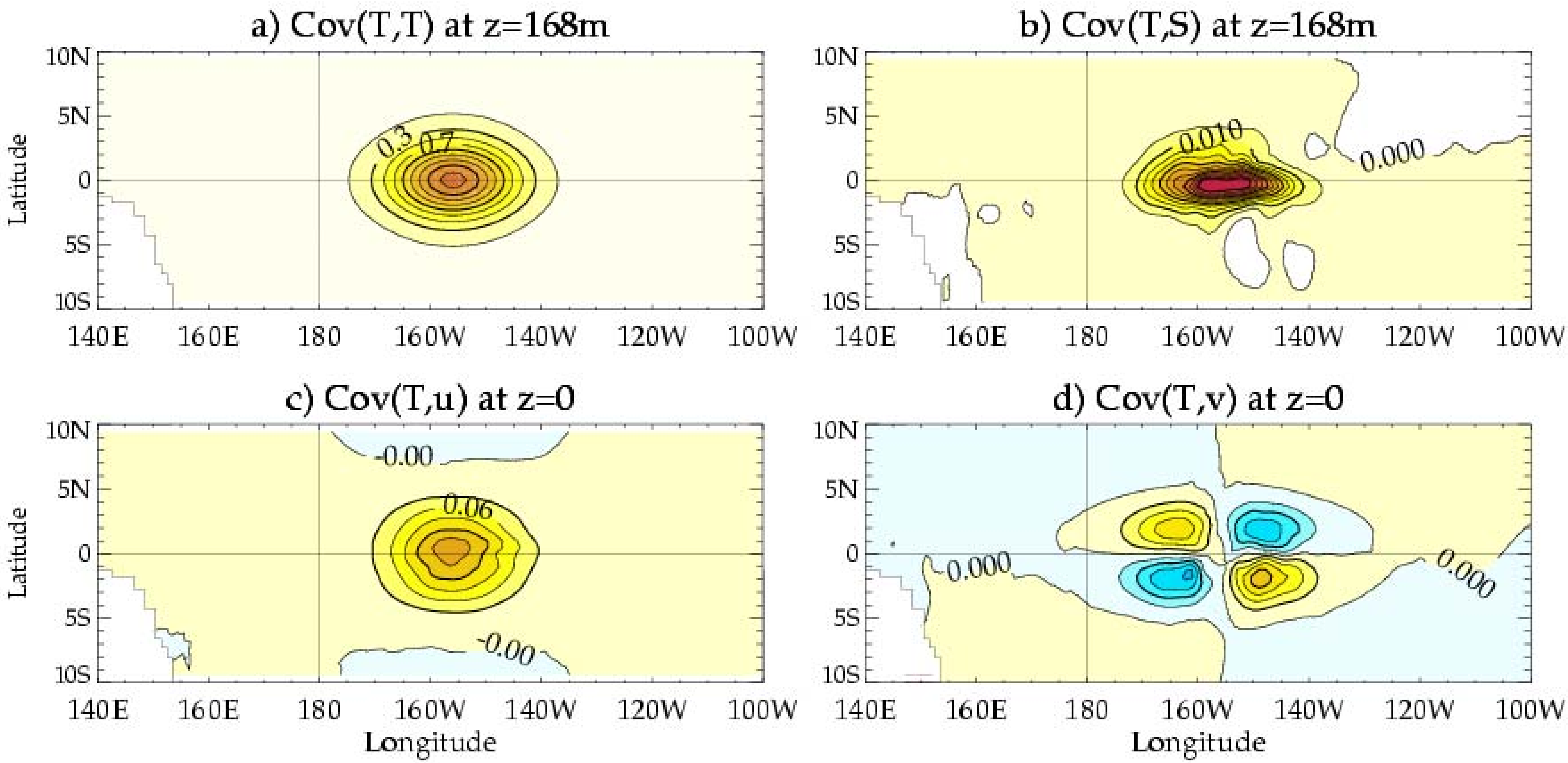
Multivariate covariance structures

<u>Example</u>: covariance relative to a SSH (η) point at (0^o,144^{o}W)



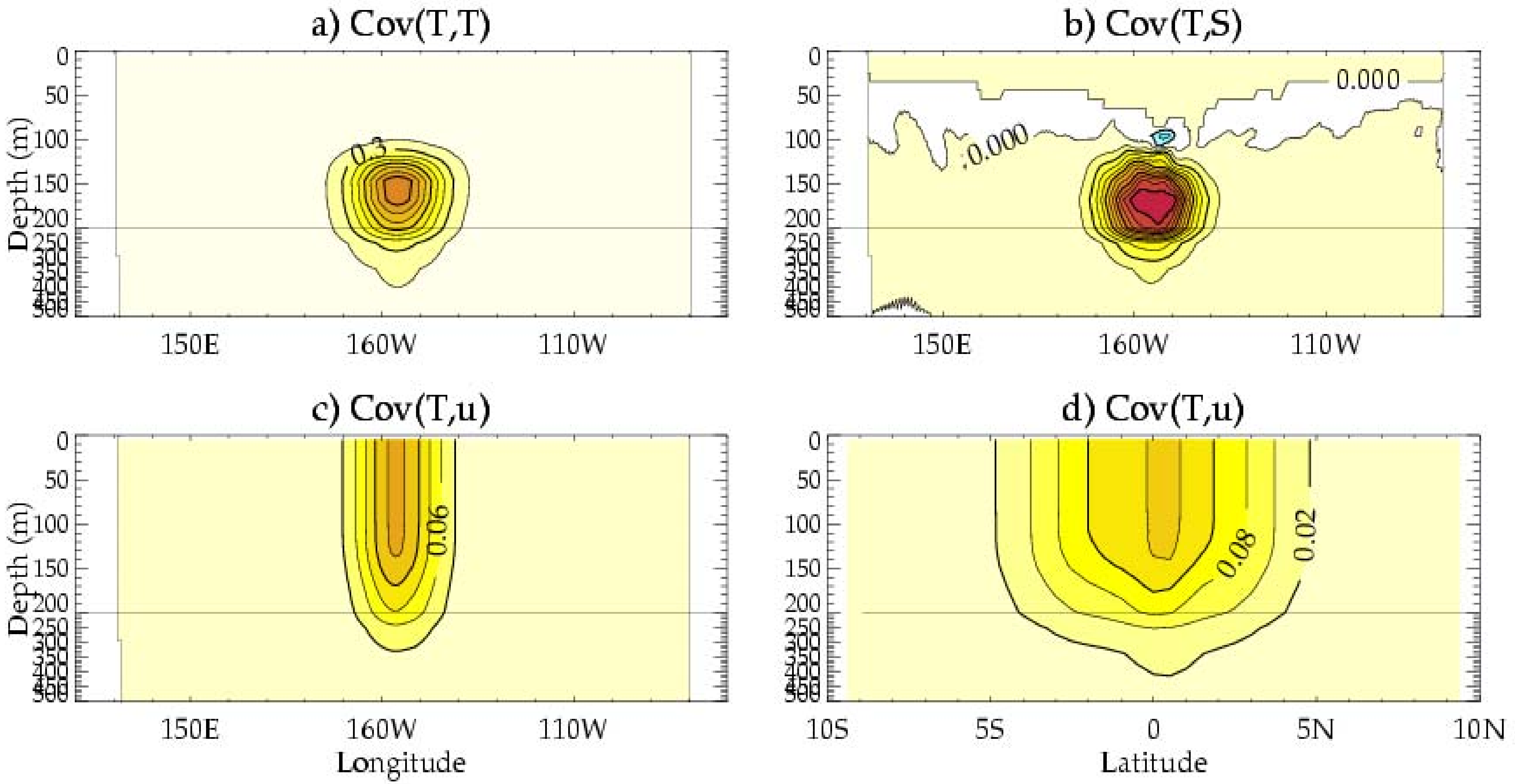
Multivariate covariance structures

<u>Example</u>: covariance relative to a T point at (0°,156°W,168m)



Multivariate covariance structures

Example: covariance relative to a T point at (0°,156°W,168m)

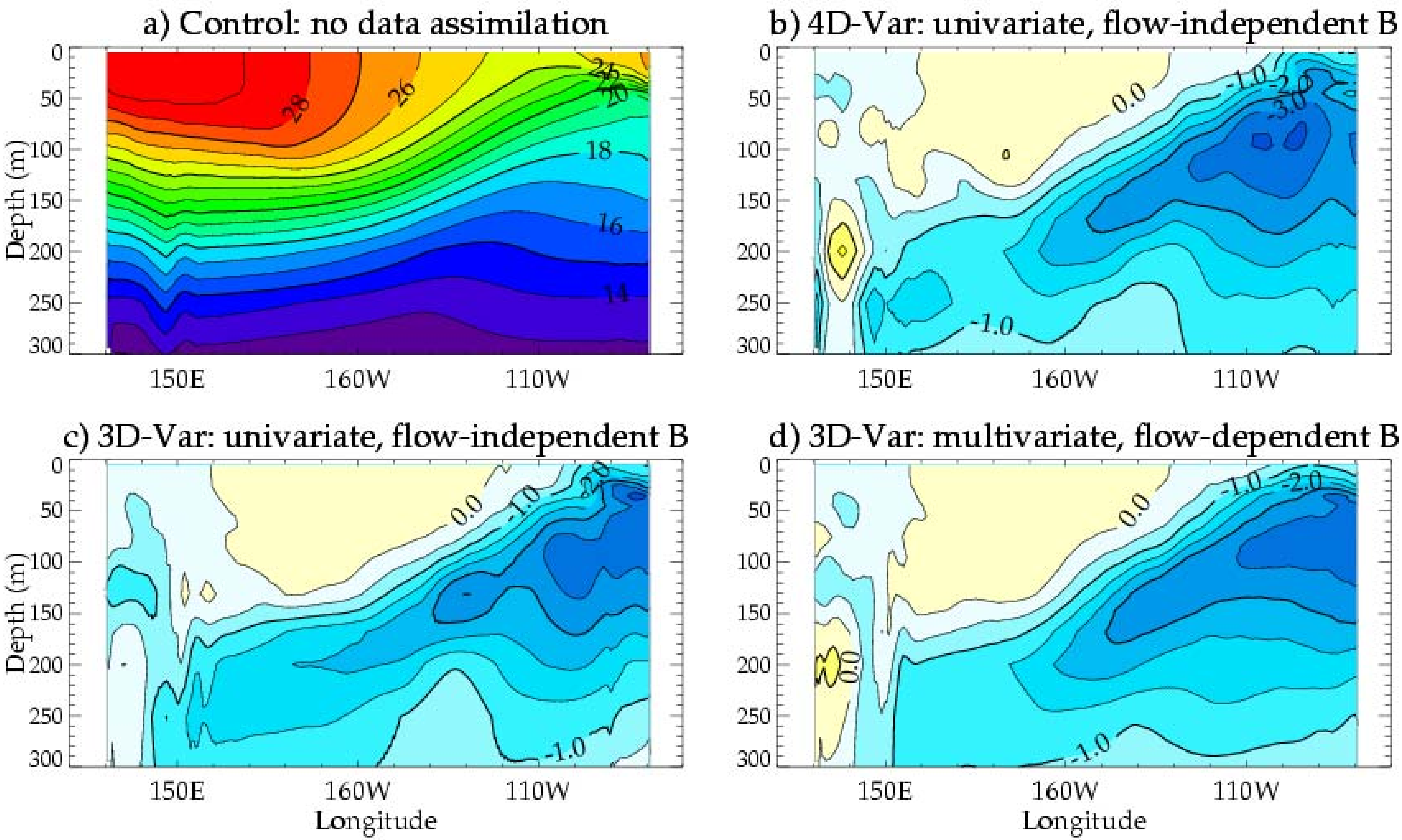


Impact on the temperature mean state

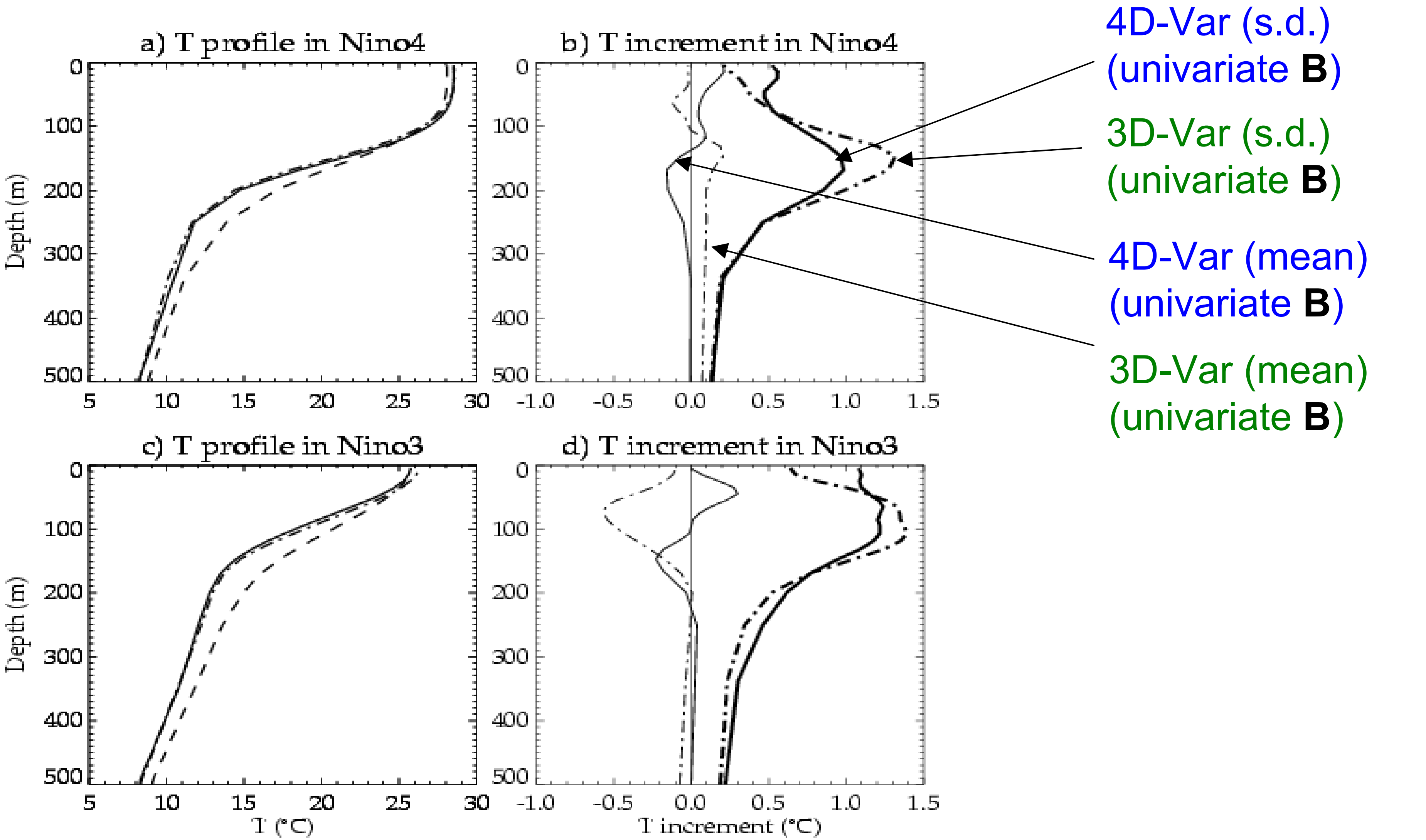
(cf. Vialard et al. 2003 - MWR)

Assimilation data-set = *in situ* temperatures from the GTSPP

1993-96 climatology: b) – d) are the difference from the control



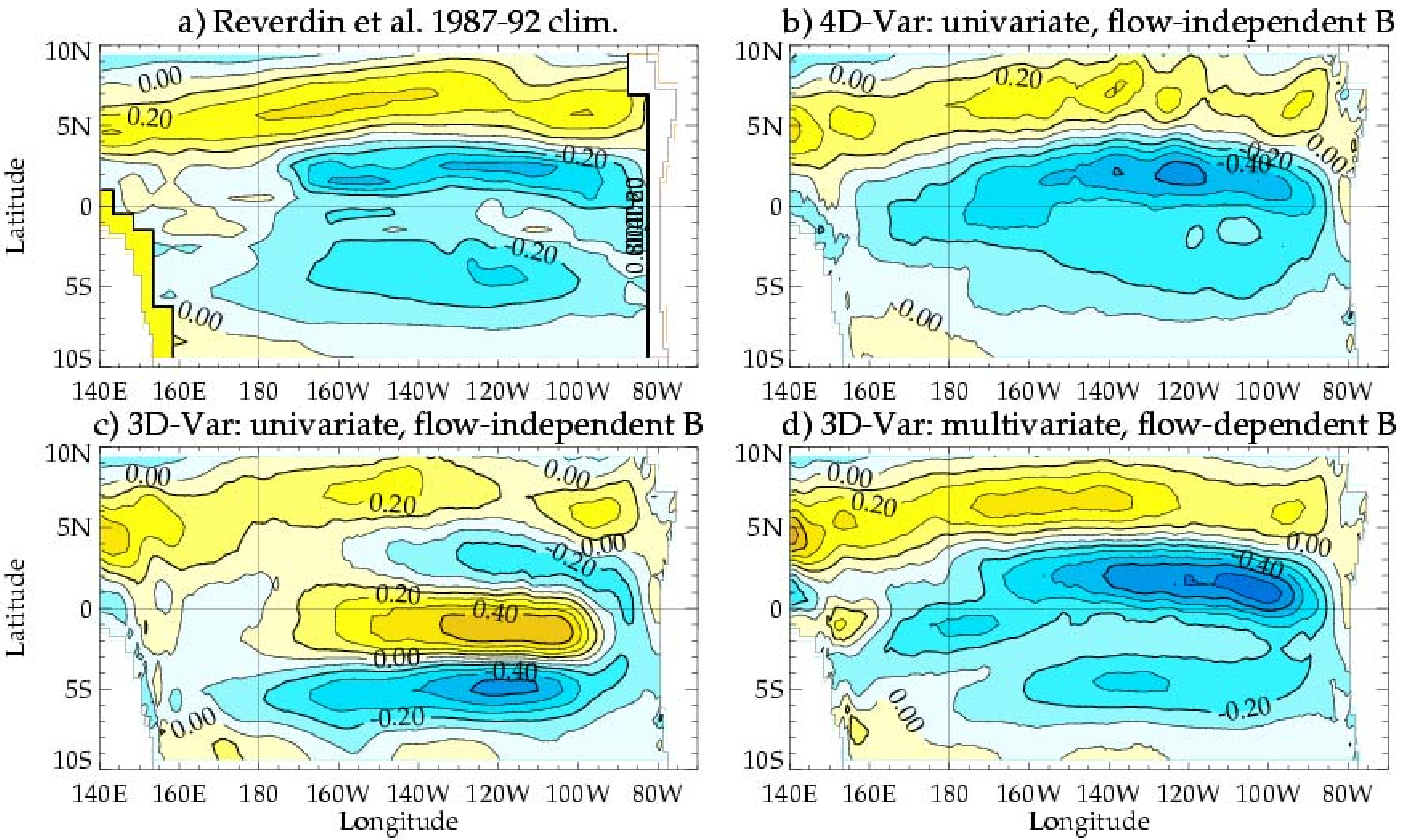
Mean and standard deviation of the analysis increments



Impact on the mean zonal velocity

(cf. Vialard et al. 2003 - MWR)

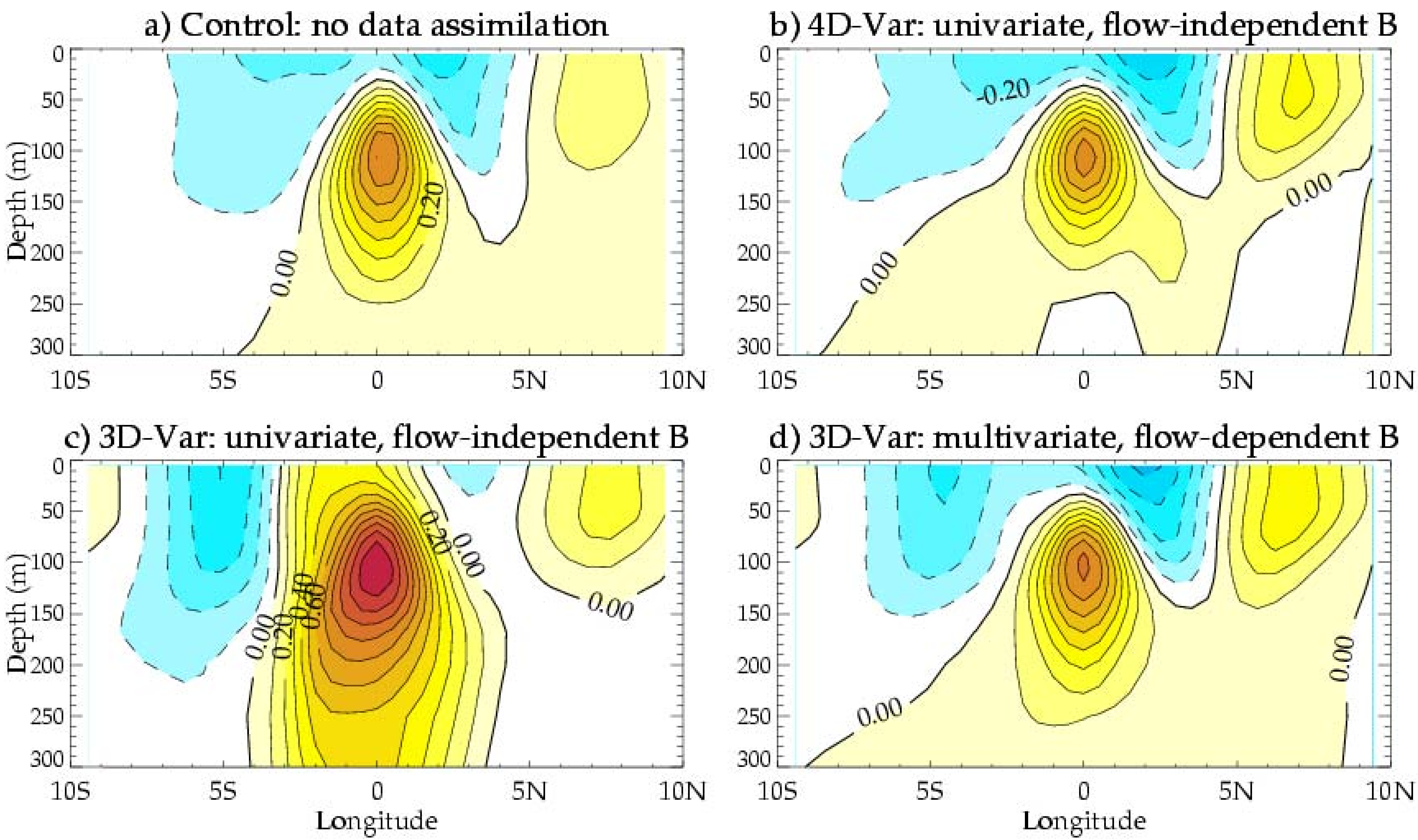
1993-96 climatology



Impact on the mean zonal velocity

(cf. Vialard et al. 2003 - MWR)

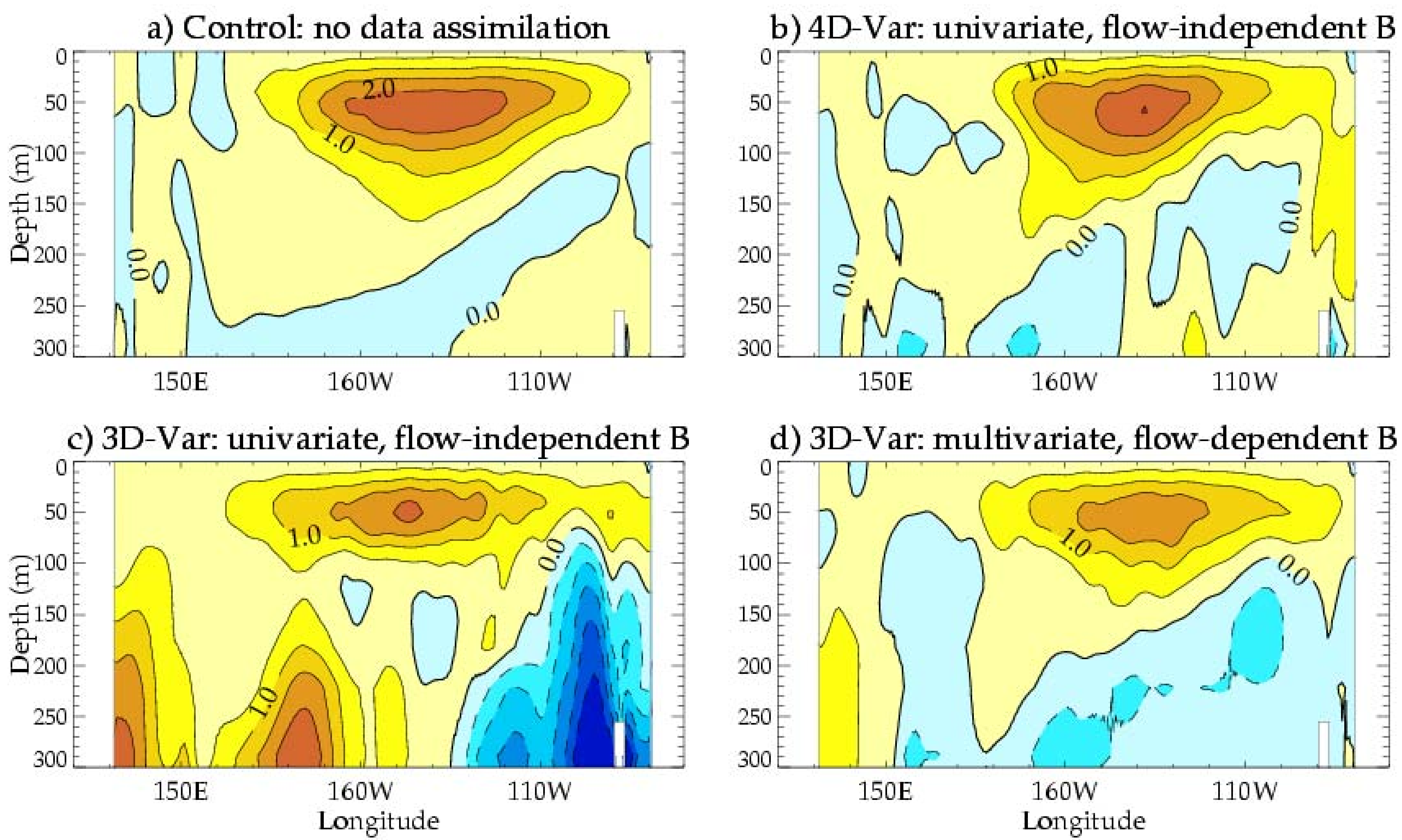
1993-96 climatology



Impact on the mean vertical velocity

(cf. Vialard et al. 2003 - MWR)

1993-96 climatology



Concluding remarks…

- Univariate data assimilation schemes tend to disrupt the dynamical balances along the equator and produce spurious circulations (*Bell et al. 2003 - QJRMS; Burgers et al. 2002 - JPO; Vialard et al. 2003 – MWR; Balmaseda lecture*).
- Improving the background error covariance models to include multivariate constraints and flow-dependent features is one way of restoring realistic balances in the model and significantly improving the analyses *(cf. Burgers et al. 2002 - JPO; Troccoli et al. 2002 – MWR; Balmaseda lecture).*
- General methodologies for modelling **B** developed in NWP are applicable to the ocean problem as well. (Ocean assimilators can exploit the wealth of experience in NWP).
- However, the details differ: need for specific algorithms for dealing with boundaries; different balance or conservation relationships, different scales,…

Research issues…

- * Develop techniques for the ocean for diagnosing the statistics of background error (ensemble methods, innovation-based methods). *(cf. Fisher lecture)*
- Develop weak constraint versions of the balance operators (* is a prerequisite).
- How do the improved covariance models benefit 4D-Var? More comparisons between 3D-Var and 4D-Var are needed.