THE USE OF DYNAMICAL KNOWLEDGE OF THE ATMOSPHERE TO IMPROVE NWP MODELS

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1 INTRODUCTION

Numerical weather prediction is carried out by integrating the complete equations of motion of the atmosphere forward from prescribed initial data. The equations include representation of a wide variety of thermodynamic effects, especially those associated with phase changes of water. Since the flow of the atmosphere cannot be completely described by a numerical model, a statistical representation of the effect of unresolved processes on the resolved flow has to be included.

Theoretical meteorology has studied the solutions of the equations of motion and thermodynamics most relevant to weather forecasting. In particular, it has been shown that most of the dry dynamics directly related to weather systems can be described by the advection of a passive scalar, the potential vorticity, from which all the other flow variables can be deduced, Hoskins et al. (1985). These ideas can be generalised to include the main effects of latent heat release and boundary layer friction.

The main job of operational models is to predict weather systems accurately. Therefore, in this chapter, we consider the design of numerical methods for the full equations of motion and thermodynamics which accurately predict the evolution of the underlying potential vorticity and control the amplitude of other motions to realistic levels.

We also consider the mathematical and numerical aspects of the sub-grid model terms, often referred to as 'physics'. The first aspect concerns the choice of a valid sub-grid model. The essence of averaging procedures is that the flow variables, when averaged over a particular scale, will vary smoothly on that scale. A valid sub-grid model will, when integrated together with the explicit equations, give solutions which are smooth over the averaging scale chosen. The second aspect concerns the interaction of the physics with the potential vorticity evolution. This needs to be understood and reflected in the way in which physical and dynamical increments are combined in the numerical method .

2 BEHAVIOUR OF THE EQUATIONS OF MOTION IN STRONGLY STRATIFIED OR STRONGLY ROTATING REGIMES

2.1 Asymptotic regimes

The technique used to analyse the equations of motion is
i) choose an asymptotic regime of interest, defined by certain parameters being small.
ii) identifying an approximation to the equations of motion which is valid for that regime.
iii) showing that this approximate system has well-behaved solutions describing the phenomena of interest.
iv) Proving that the solution of the complete equations stays close to that of the approximate system (and converges to it as the small parameters tend to zero).

In the case of extra-tropical weather systems, the regimes of interest are where either the rotation or the stratification is strong. The approximate equations can be written in terms of the advection of potential vorticity by a velocity field deduced from the potential vorticity. Step (iv) allows the amplitude of inertia-gravity waves to be estimated. The same approach can be used to study other phenomena, such as organised convection, with an appropriate but different choice of small parameters.
We illustrate this using the hydrostatic equations in pressure coordinates in the vertical and Cartesian coordinates in the horizontal. This is the simplest system in which the necessary points can be made. This material is based on the analysis of Warn et al. (1995) and Vallis (1996). More background material is contained in standard textbooks,

e.g. Haltiner and Williams (1980). The notation is standard except as indicated.

$$\begin{aligned}
\frac{Du}{Dt} + \frac{\partial\phi}{\partial x} - fv &= 0 \\
\frac{Dv}{Dt} + \frac{\partial\phi}{\partial y} + fu &= 0 \\
\frac{\partial\phi}{\partial p} &= \frac{\theta}{\hat{\theta}}\frac{\partial\hat{\phi}}{\partial p} \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial\omega}{\partial p} &= 0 \\
\frac{D\theta}{Dt} + \omega\frac{\partial\hat{\theta}}{\partial p} &= 0 \\
\omega \equiv \frac{Dp}{Dt} &= 0 \text{ at } p = 0 \\
\omega = \frac{\partial p_s}{\partial t} + \mathbf{u}_h \cdot \nabla p_s &\text{ at } p = p_s \\
\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} &+ \omega\frac{\partial}{\partial p}
\end{aligned} \tag{1}$$

The lower boundary condition at $p = p_s$ is that $p = p_s$ is a material surface. $\hat{\theta}$ is a basic state which is a function of p only. The equivalent basic state geopotential satisfies $\partial\hat{\phi}/\partial p = R\hat{\theta}\Pi/p$, where Π is the Exner pressure and R the gas constant.

These equations imply the following potential vorticity conservation law

$$\frac{DQ}{Dt} = 0 \tag{2}$$

$$Q = \left(f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\left(\frac{\partial\hat{\theta}}{\partial p} + \frac{\partial\theta}{\partial p}\right) - \left(\frac{\partial v}{\partial p}\frac{\partial\theta}{\partial x} - \frac{\partial u}{\partial p}\frac{\partial\theta}{\partial y}\right)$$

In order to understand the effect of the boundary conditions, and the behaviour of large scale flow, it is of interest to consider the special case where $\hat{\theta}$ takes a uniform value θ_0, $\theta = 0$, and u and v are independent of p. Then

$$\begin{aligned}
\frac{Du}{Dt} + \frac{\partial\phi}{\partial x} - fv &= 0 \\
\frac{Dv}{Dt} + \frac{\partial\phi}{\partial y} + fu &= 0 \\
\phi &= C_p\theta_0\Pi_s \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{p_s}\frac{Dp_s}{Dt} &= 0
\end{aligned} \tag{3}$$

This is equivalent to a shallow water system. The potential vorticity is

$$Q = \left(f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) / p_s \tag{4}$$

The key parameters determining the nature of the flow are the Rossby number Ro, the Froude number Fr and the Burger number Bu, where

$$\begin{aligned} Ro &= U/fL \text{ Rossby number} \\ Fr &= U/NH \text{ Froude number} \\ Bu &= Ro/Fr \text{ Burger number} \end{aligned} \tag{5}$$

Here U, L, H are scales for horizontal velocity, horizontal length scale and vertical length scale, and N^2 is the Brunt-Vaisala frequency $\frac{1}{\theta}\frac{\partial\hat{\theta}}{\partial p}$. In the 'shallow water' case (3), the Froude number is U/c, where $c^2 = R\theta_0$. The scalings associated with weather systems are that at least one of Ro and Fr is small, and that H/L is small. Other situations such as active convection have different scalings. Baroclinic instability has $Bu = 1$ with both Ro and Fr small. In particular the 'Charney' scaling sets $Ro = Fr$, so $Bu = 1$, and then lets them become small together. In this chapter we consider the cases either Ro or Fr small, with the other $\leq O(1)$, and Bu taking any value. This covers the spectrum from strongly rotating, weakly stratified flow, to the converse. It is often convenient to express this, for fixed H, N and F, as a choice between $L > L_R$ and $L < L_R$; where L_R, the Rossby radius of deformation, is NH/f.

2.2 Approximate solutions

It is convenient to analyse equations (1) by writing them in terms of the potential vorticity equation (2), an equation for the horizontal divergence χ and an equation for the 'linear imbalance', defined below. This is similar to the procedures used by Vallis (1996) and also Lynch (1989) and Temperton (1988) to define 'slow' equations or 'implicit normal mode' initialisation.

Let ∇ represent the three dimensional operator $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial p})$ and ∇_h the two-dimensional operator $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0)$, with similar meanings of the subscript h for other vector operators. Let $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ be the vertical component of the relative vorticity and $\chi = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ the horizontal divergence. To preserve generality in the definition of the Cartesian coordinates we allow for f to vary in both x and y. Then the divergence equation is

$$\frac{D\chi}{Dt} + \chi^2 + 2J(u,v) + \frac{\partial\omega}{\partial x}\frac{\partial u}{\partial p} + \frac{\partial\omega}{\partial y}\frac{\partial v}{\partial p} - f\zeta - v\frac{\partial f}{\partial x} + u\frac{\partial f}{\partial y} + \nabla_h^2\phi = 0 \qquad (6)$$

The 'imbalance' equation is

$$\xi = \frac{1}{\hat\theta}\frac{\partial\hat\phi}{\partial p}\nabla_h^2\theta - f\frac{\partial\zeta}{\partial p}$$

$$\begin{aligned}\frac{\partial\xi}{\partial t} = &-\frac{1}{\hat\theta}\frac{\partial\hat\phi}{\partial p}\nabla_h^2(\mathbf{u}\cdot\nabla\theta) - N^2\nabla_h^2\omega + f\frac{\partial}{\partial p}(\mathbf{u}\cdot\nabla(\zeta+f)) \\ &-f(\zeta+f)\frac{\partial^2\omega}{\partial p^2} + f\left(\chi\frac{\partial\zeta}{\partial p} + \frac{\partial}{\partial p}\left(\frac{\partial\omega}{\partial x}\frac{\partial v}{\partial p} - \frac{\partial\omega}{\partial y}\frac{\partial u}{\partial p}\right)\right)\end{aligned} \qquad (7)$$

The continuity equation and the definition of N^2 have been used to simplify this equation. Note that ξ is not the true residue in the linear balance equation because the derivatives of f have not been included in the definition of ξ. The equation for the latter is a lot more complicated. Equation (7) is simpler for scale analysis.

Substitute (6) into (7). This gives

$$\begin{aligned}\frac{\partial}{\partial t^2}\frac{\partial^2\omega}{\partial p^2} + \frac{\partial^2}{\partial t\partial p}\left(-\mathbf{u}\cdot\nabla\chi - \chi^2 - 2J(u,v) + \cdots\right) = \\ -\left(N^2\nabla_h^2\omega + f(\zeta+f)\frac{\partial^2\omega}{\partial p^2}\right) + f\left(\chi\frac{\partial\zeta}{\partial p} + \frac{\partial}{\partial p}\left(\frac{\partial\omega}{\partial x}\frac{\partial v}{\partial p} - \frac{\partial\omega}{\partial y}\frac{\partial u}{\partial p}\right)\right) \\ -\frac{1}{\hat\theta}\frac{\partial\hat\phi}{\partial p}\nabla_h^2(\mathbf{u}\cdot\nabla\theta) + f\frac{\partial}{\partial p}(\mathbf{u}\cdot\nabla(\zeta+f))\end{aligned} \qquad (8)$$

We now estimate the size of the divergence χ, or equivalently $\frac{\partial\omega}{\partial p}$. The horizontal momentum equations and the hydrostatic relation from (1) show that

$$\nabla_h\theta = O\left(\frac{fU(1+Ro)}{N^2H}\frac{\partial\hat\theta}{\partial p}\right) \qquad (9)$$

Seek solutions with χ is small, and the potential vorticity having timescale U/L. First omit the term $\frac{\partial\omega}{\partial t}$ from (8). This gives

$$\begin{aligned}\frac{\partial^2}{\partial t\partial p}(-2J(u,v) + \cdots) = \\ -\left(N^2\nabla_h^2\omega + f(\zeta+f)\frac{\partial^2\omega}{\partial p^2}\right) + f\left(\chi\frac{\partial\zeta}{\partial p} + \frac{\partial}{\partial p}\left(\frac{\partial\omega}{\partial x}\frac{\partial v}{\partial p} - \frac{\partial\omega}{\partial y}\frac{\partial u}{\partial p}\right)\right) \\ -\frac{1}{\hat\theta}\frac{\partial\hat\phi}{\partial p}\nabla_h^2(\mathbf{u}\cdot\nabla\theta) + f\frac{\partial}{\partial p}(\mathbf{u}\cdot\nabla(\zeta+f))\end{aligned} \qquad (10)$$

The terms in the bracket on the left hand side of (10) are of order $U^2/(HL^2) + \beta U/H$, where β is a typical rate of variation of f. The left hand side of (10) is therefore of order $\frac{U^3}{HL^3} + \beta \frac{U^2}{HL}$. The right hand side can be estimated as

$$-\omega \left(N^2/L^2 + f^2(1+Ro)/H^2\right) + \frac{fU^2}{HL^2} + \frac{\beta f U}{H} \tag{11}$$

Thus we can estimate

$$\omega \leq \frac{UH}{L} \frac{RoFr^2}{Ro^2 + Fr^2} \tag{12}$$

(if β is no greater than Rof/L). The three cases of interest are the strongly stratified regime $Ro = O(1), Fr \ll 1$, the 'Charney' scaling $Ro = Fr \ll 1$, and the strongly rotating regime $Fr = O(1), Ro \ll 1$. (12) shows that $\omega \simeq \epsilon \frac{UH}{L}$, where ϵ is respectively $O(Fr^2), O(Ro)$, and $O(Ro)$ in the three cases.

(8) shows that the timescale of inertio-gravity waves is $O\sqrt{(}N^2H^2/L^2 + f^2$. Assume that this is the timescale of ω. Then a self-consistent solution of (1) will be a flow determined by potential vorticity advection, with superposed inertio-gravity waves of amplitude $O(\epsilon)$, where ϵ takes the values above in the three regimes. The fast timescale of motions of amplitude $O(\epsilon)$ will not alter the timescale U/L of the overall flow. The effect of the waves on the potential vorticity evolution will be at most $O(\epsilon)$, but time averaging over the wave period can be used to show that the real interaction is much weaker (Babin et al. (1996,1997)). They also show that the inertio-gravity waves cascade efficiently to small scales if $Bu \geq 1$, so that the total flow will be closer to the solution of (10) in that case than our analysis suggests. In the case $Bu \ll 1$, the rotation dominated case, the cascades are inefficient and the $O(Ro)$ estimate of inertio-gravity wave amplitudes is realistic. Observations confirm the prevalence of quasi-inertial waves in the atmosphere on small vertical scales, corresponding to $Bu \ll 1$, (Sidi and Barat (1986)). The sensitivity of this cascade to Bu has to be allowed for in desiging schemes which damp inertio-gravity waves.

In numerical methods for equation (1), the need is to control the amplitude of inertio-gravity waves to $O(\epsilon)$, and to ensure that the part of the potential vorticity evolution which can be predicted independently of the inertio-gravity waves is accurately treated. The latter is determined by using the divergence calculated from (10), where the term in $\frac{\partial u}{\partial t}$ on the left hand side is substituted for using the rotational part of the momentum equations from (1). The numerical method must therefore contain an accurate analogue of (10). If both real and simulated inertio-gravity waves time average to zero, then the total error in the potential vorticity advection will be the error in approximating (10), together with the error in parametrizing the higher order interaction of the inertio-gravity waves on the potential vorticity field.

These estimates show why the solution of (1) in a regime where Ro or Fr is small can be almost exactly replicated by that of a system where the divergence is diagnosed implicitly, and the evolution governed by potential vorticity advection. This has been demonstrated many times, for instance by Allen et al. (1990), and McIntyre and Norton (1998). In particular, if the initial amplitude of inertio-gravity waves is zero, the potential vorticity with associated divergence calculated from (10) is a solution of (1) to order Fr^4, Ro^3 and Ro^3 in the three regimes. This follows from assuming a timescale U/L for the term in $\partial^2\omega/\partial t^2$ in (8), whence the ratio of this term to the terms on the right hand side is ϵRo. Therefore the difference between a solution of (10) and a solution of (1) is $O(\epsilon^2 Ro)$. In the case $Ro = O(1), Fr \ll 1$ this is consistent with the estimate made by Ford et al. (1998) of the effect of spontaneous gravity wave emission on the potential vorticity evolution.

The potential vorticity equation (2) is derived from the vorticity and thermodynamic equations derived from (1). (2) holds whatever the value of the divergence χ, and vertical motion deduced from it. The potential vorticity (4) associated with the vertically meaned flow is derived from the vorticity and surface pressure equation, again for arbitrary χ. Therefore, in a numerical method, accurate treatment of the advection of ζ, θ and p_s, together with a consistent definition of χ, and an appropriate method of estimating it as discussed above, will ensure potential vorticity conservation.

2.3 Inclusion of physical effects

In addition to the terms included in (1), operational models include extra terms in the momentum and thermodynamic equations normally lumped together as 'physics'. These can be described as either

i) Direct source terms (e.g. radiation).

ii) Boundary flux terms.

iii) Terms representing phase changes.

iv) Terms representing sub-gridscale dynamics.

(iv) includes vertical and horizontal diffusion, boundary layer mixing, gravity wave drag, and much of the deep and shallow convection schemes (except phase changes). In addition, all the other terms have to allow for sub-gridscale variability.

If (1) are regarded formally as equations for the averaged flow, where an overbar represents a general space-time filtering operation, we obtain

$$\begin{aligned}
\frac{D\overline{u}}{Dt} + \frac{\partial\overline{\phi}}{\partial x} - f\overline{v} &= F_u \\
\frac{D\overline{v}}{Dt} + \frac{\partial\overline{\phi}}{\partial y} + f\overline{u} &= F_v \\
\frac{\partial\overline{\phi}}{\partial p} &= \frac{\overline{\theta}}{\hat{\theta}}\frac{\partial\phi}{\partial p} \\
\frac{\partial\overline{u}}{\partial x} + \frac{\partial\overline{v}}{\partial y} + \frac{\partial\overline{\omega}}{\partial p} &= 0 \\
\frac{D\overline{\theta}}{Dt} + \overline{\omega}\frac{\partial\hat{\theta}}{\partial p} &= F_\theta \\
\overline{\omega} \equiv \frac{Dp}{Dt} &= 0 \text{ at } p = 0 \\
\overline{\omega} = \frac{\partial\overline{p}_s}{\partial t} + \overline{\mathbf{u}}_h\cdot\nabla\overline{p}_s + F_p &\text{ at } p = \overline{p}_s \\
\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x} + \overline{v}\frac{\partial}{\partial y} &+ \overline{\omega}\frac{\partial}{\partial p}
\end{aligned} \tag{13}$$

The terms F_u, F_v, F_θ, F_p represent the sub-grid model (iv). The unaveraged equations (1) can generate small scales and turbulent solutions. The basic flow variables u, v, θ, p_s are bounded, and so their average values will vary smoothly over the filter scale. Thus, if the sub-grid model is correctly designed, the solutions of (13) will also vary smoothly over the filter scale. An example of a standard method that does not meet this criterion is stability-dependent vertical diffusion of potential temperature. This can be written

$$\frac{\partial\overline{\theta}}{\partial t} = \frac{\partial}{\partial p}K\frac{\partial\overline{\theta}}{\partial p} \tag{14}$$

where $K \to 0$ as $\partial\theta/\partial p \to -\infty$, and $K \to \infty$ as $\partial\theta/\partial p \to 0$. The analytic solution of this equation is that θ is a piecewise constant function of p, on an arbitrarily small scale determined by the variations in $\partial\theta/\partial p$ in the initial data. This clearly does not satisfy any smoothness requirement.

A more extreme example is that of a super-adiabatic boundary layer. Equations (1) will have unstable solutions in this case corresponding to small scale convection. Equations (13), without the sub-grid terms, are formally identical, and so will also have unstable solutions. The sub-grid model must suppress this instability, because observations show that super-adiabatic boundary layers can persist for long periods in the averaged state if the forcing from below is strong enough. While sufficiently high horizontal diffusion can ensure this, it may not be the most realistic sub-grid model. Another option is to replace the static stability in teh expolicit terms by a neutral value, and replace the transports by a non-local parametrization. If the instability results from moisture, the best solution may be to disable the large scale precipitation calculation and only use the convection scheme to calculate the transports.

Another example of an invalid sub-grid model is a convection scheme whose interaction with the dynamics produces grid-scale convection. In general the smoothness criterion will often be failed by schemes with logical switches depoending on grid-scale detail of the flow. On the other hand, linear diffusion will satisfy smoothness criteria, but lose physical realism.

2.4 Interaction of physics with the potential vorticity evolution

We now show how physical effects can couple to the potential vorticity evolution, and hence the evolution of weather systems. This is done by generalising the analysis of section 2.2.

It is useful to start from an alternative method of analysing (1), based on Schubert (1985). Write (1) in the form

$$\mathbf{Q}\begin{pmatrix} u \\ v \\ \omega \end{pmatrix} + \frac{\partial}{\partial t}\begin{pmatrix} fv \\ -fu \\ \frac{1}{\theta}\frac{\partial\hat{\phi}}{\partial p}\theta \end{pmatrix} = \mathbf{H} \tag{15}$$

where

$$\mathbf{Q} = \begin{pmatrix} f\left(\frac{\partial v}{\partial x} + f\right) & f\frac{\partial v}{\partial y} & f\frac{\partial v}{\partial p} \\ -f\frac{\partial u}{\partial x} & f\left(f - \frac{\partial u}{\partial y}\right) & -f\frac{\partial u}{\partial p} \\ \frac{1}{\theta}\frac{\partial \hat{\phi}}{\partial p}\frac{\partial \theta}{\partial x} & \frac{1}{\theta}\frac{\partial \hat{\phi}}{\partial p}\frac{\partial \theta}{\partial y} & \frac{1}{\theta}\frac{\partial \hat{\phi}}{\partial p}\frac{\partial \theta}{\partial p} \end{pmatrix} \tag{16}$$

and

$$\mathbf{H} = \begin{pmatrix} -f\frac{\partial \phi}{\partial y} \\ f\frac{\partial \phi}{\partial x} \\ 0 \end{pmatrix} \tag{17}$$

Instead of neglecting $\frac{\partial \chi}{\partial t}$ as in section 2.2, we replace $\frac{\partial}{\partial t}(u, v)$ by its geostrophic value. If f is constant, this is equivalent to neglecting both $\frac{\partial \chi}{\partial t}$ and $\frac{\partial \xi}{\partial t}$. (10) shows that this will give an $O(Ro)$ estimate of ω if $Ro \ll 1$. It does not give a useful approximation if $Ro = O(1)$. We then obtain

$$\mathbf{Q}\begin{pmatrix} u \\ v \\ \omega \end{pmatrix} + \frac{\partial}{\partial t}\nabla\phi = \mathbf{H} \tag{18}$$

(18) can be interpreted as $\mathbf{H}$ being the dynamical forcing, $\mathbf{Q}$ determines the response of the atmosphere to forcing, and $\frac{\partial}{\partial t}$ is the evolution of the geostrophic pressure, which will be related to the potential vorticity. It can be shown that det $\mathbf{Q}$ approximates the potential vorticity Q to $O(Ro)$. Thus small potential vorticity implies that $\mathbf{Q}$ has at least one small eigenvalue, and so there will be at least one direction where small forcing can generate a large response.

We now generalise this formulation to include physical effects. In the boundary layer, geostrophic balance is replaced by Ekman balance, defined by

$$\begin{aligned} \frac{\partial \phi}{\partial x} - f v_e &= F(u_e) \\ \frac{\partial \phi}{\partial y} + f u_e &= F(v_e) \end{aligned} \tag{19}$$

Assume first that the friction terms $F(u), F(v)$ can be represented as a linear drag $-c_D(u, v)$. Set up an equation similar to (15), by forming a linear combination of the first two equations of (13). Replace potential temperature gradients by equivalent potential temperature gradients in saturated regions. We then obtain

$$\mathbf{Q}\begin{pmatrix} u \\ v \\ \omega \end{pmatrix} + \frac{\partial}{\partial t}\begin{pmatrix} fv - c_D u \\ -fu - c_D v \\ \frac{1}{\hat{\theta}}\frac{\partial \hat{\phi}}{\partial p}\theta \end{pmatrix} = \mathbf{H} \tag{20}$$

where

$$\mathbf{Q} = \begin{pmatrix} f\frac{\partial v}{\partial x} - c_D\frac{\partial u}{\partial x} + f^2 + c_D^2 & f\frac{\partial v}{\partial y} - c_D\frac{\partial u}{\partial y} & f\frac{\partial v}{\partial p} - c_D\frac{\partial u}{\partial p} \\ -f\frac{\partial u}{\partial x} - c_D\frac{\partial v}{\partial x} & f^2 + c_D^2 - f\frac{\partial u}{\partial y} - c_D\frac{\partial v}{\partial y} & -f\frac{\partial u}{\partial p} - c_D\frac{\partial v}{\partial p} \\ \frac{1}{\hat{\theta}}\frac{\partial \hat{\phi}}{\partial p}\frac{\partial \Theta}{\partial x} & \frac{1}{\hat{\theta}}\frac{\partial \hat{\phi}}{\partial p}\frac{\partial \Theta}{\partial y} & \frac{1}{\hat{\theta}}\frac{\partial \hat{\phi}}{\partial p}\frac{\partial \Theta}{\partial p} \end{pmatrix} \tag{21}$$

and

$$\mathbf{H} = \begin{pmatrix} -f\frac{\partial \phi}{\partial y} + c_D\frac{\partial \phi}{\partial x} \\ f\frac{\partial \phi}{\partial x} + c_D\frac{\partial \phi}{\partial y} \\ S_h \end{pmatrix} \tag{22}$$

$\nabla\Theta$ is the gradient of the equivalent potential temperature in saturated regions and the actual potential temperature otherwise. S_h represents source terms in the thermodynamic equation. If we now assume that the time derivatives of u and v obey Ekman balance, we recover (18).

The structure of equation (18) with (21) and (22) is again that the forcing terms are on the right hand side and the response to the forcing is on the left hand side. The response is a velocity field $\mathbf{u}$ which is largest in the direction of the eigenvector of $\mathbf{Q}$ with the smallest eigenvalue. We can see that boundary layer friction enters both sides of (20). The diagonal terms of $\mathbf{Q}$ are similar in magnitude to those in (15) but with f^2 replaced by $f^2 + c_D^2$ and the static stability replaced by its moist value in saturated regions. Thus the effect of friction is to increase the eigenvalues of $\mathbf{Q}$, and thus stabilise the atmosphere against forcing. In particular, it will reduce the effective Burger number, and hence the Rossby radius of deformation. The effect of latent heating is to make the atmosphere more responsive to forcing rather than being a forcing term in itself.

If a more complete formulation of boundary layer friction is used, the matrix $\mathbf{Q}$ will become non-local. However, the stabilising effect is likely to remain. Similarly, the effect of precipitation will be to make a non-local change to $\mathbf{Q}$, if the precipitation rate is assumed to be essentially proportional to the rate of generation of supersaturation. However, this could be stabilising or destabilising depending on the circumstances, and the assumptions made in deriving (18) are only likely to be valid for quasi-steady systems.

When applying this thinking to operational models, the idea is to ensure the coupling implied by (18) if the timestep is long. This will involve coupling implicit treatment of the boundary layer with implicit treatment of the dynamics. It also involves replacing potential temperature gradients with equivalent potential temperature gradients in implicit calculations in saturated regions. In practice, the easiest way to implement this is to use a standard parametrization for the explicit update, and a linearised correction to it in the implicit step. Thus we could write

$$\begin{aligned} u^{n+1} &= u^n + F_u^n + c_D u' \\ v^{n+1} &= v^n + F_v^n + c_D v' \end{aligned} \tag{23}$$

This treats the boundary layer drag with a possibly complex non-local scheme in the explicit step, but a linearised correction in the form of an equivalent drag in the implicit step. Similar linearisations are needed for incremental 4DVAR.

If there is actual convective instability, $\mathbf{Q}$ will have a negative eigenvalue, and the response to forcing in terms of the balanced dynamics will be discontinuous mass transport, Shutts (1995). In practice, this is represented by a parametrization scheme. The necessary coupling expressed by (18) can only be achieved by treating the convective mass transport implicitly. This can be done by using the full, complex scheme in the explicit step, and a linearised correction in terms of a perturbation to the mass flux in the implicit step. This will allow a change in forcing to produce a response in terms of more or less convection, rather than in changes to large scale vertical motion. Such a scheme would be more expensive, because the finite difference representation of $\mathbf{Q}$ would become non-local in the vertical. However, the real processs is non-local and this may be unavoidable if realistic results are required. A similar situation will arise if precipitation effects are strongly coupled to the balanced dynamics. Non-local implicit methods will have to be used if correct behaviour is to be obtained with a large timestep.

The analysis of section 2.2 can be extended to give some indications as to when physical processes will strongly couple to the potential vorticity evolution, and when the response would be largely unbalanced. This can be done by replacing f by $\sqrt{(f^2 + c_D^2)}$ and N by its moist value in the definitions of Ro and Fr.

Outside the boundary layer the frictionless estimates for the magnitude of ω can still be obtained by using the Ekman pumping velocity at the boundary layer top as a lower boundary condition, instead of the dynamical condition contained in (1). The frictionless scaling will be unaffected if the ratio of the boundary layer depth to the vertical scale of the motion is no greater than ϵ, with the appropriate choice of ϵ for the asymptotic regime.

Localised thermal forcing will enter directly on the right hand side of (7). A forcing S will generate a response given approximately by

$$N^2\nabla^2\omega + f(\zeta + f)\frac{\partial^2\omega}{\partial p^2} = \frac{1}{\hat{\theta}}\frac{\partial\hat{\phi}}{\partial p}\nabla_h^2 S \qquad (24)$$

The scalings remain valid if the resulting $\omega \leq \epsilon UH/L$, so that $S \leq \frac{fU}{N^2H}\frac{\partial\hat{\theta}}{\partial p}$.

2.5 Qualitative behaviour of potential vorticity evolution

The analysis above suggests that typical behaviour can be described in terms of two cases. When $Fr \ll 1, Ro = O(1)$, gravity waves are fast and there is a potential vorticity based solution which is almost free of gravity waves. When $Ro \ll 1$, there is a basic flow close to geostrophic balance, together with inertia-gravity waves with amplitude $O(Ro)$ times that of the basic flow.

We now show that the qualitative behaviour of the horizontal flow changes according to whether L is greater or less than L_R, the Rossby radius. For a mid-latitude values of the Coriolis parameter, 10^{-4}, and a typical model vertical structure, the Rossby radius associated with the external mode is about 3000km (global wavenumber 2), with the 3rd internal mode about 1000km (wavenumber 6) and with the 10th internal mode about 200km (wavenumber 30). These normal mode values are influenced by the strong stability of the stratosphere. A typical tropospheric value of N^2 would give $L_R \simeq 1000$km if the depth scale is the depth of the troposphere. The maximum baroclinic instability tends to be on a scale of about 2000km, which is the effective Rossby radius in the region where it occurs.

We can illustrate this most simply by studies of the quasi-geostrophic approximation to the potential vorticity equation (2). If f and N^2 are constant, this is

$$\frac{\partial Q}{\partial t} + \mathbf{u}_g \cdot \nabla Q = 0 \tag{25}$$
$$Q = (f + f^{-1}\nabla^2\phi)\left(\frac{N^2 p}{R\Pi} + \frac{\partial}{\partial p}\left(\frac{p}{R\Pi}\frac{\partial\phi}{\partial p}\right)\right)$$

Larichev and McWilliams (1991) and Farge and Sadourny (1991) studied the behaviour of this equation in both regimes. In the case $L \ll L_R$, the variations of Q are dominated by variations in relative vorticity. Thus (25) behaves like

$$\frac{\partial\nabla^2\phi}{\partial t} + \mathbf{u}_g \cdot \nabla(\nabla^2\phi) = 0 \tag{26}$$
$$(fv_g, -fu_g) = \nabla_h\phi$$

This is exactly the equation for two-dimensional incompressible flow. The solutions produce a cascade of energy to large scales and enstrophy to small scales. This identification is reponsible for statements that the large scale flow of the atmosphere is like two-dimensional turbulence (Leith (1981)). However, this requires $L \leq L_R$, which is not strictly large scale flow. Observed filamentation of the stratospheric vortex may satisfy $L \leq L_R$, though the height scale for this type of motion is uncertain. In the case $L \gg L_R$, (25) behaves like

$$\frac{\partial}{\partial t}\frac{\partial^2\phi}{\partial p^2} + \mathbf{u}_g \cdot \nabla\frac{\partial^2\phi}{\partial p^2} = 0 \tag{27}$$

The behaviour of this equation has not been studied. However, if the effect of the operator $\partial/\partial p^2$ is represented by multiplying by a vertical eigenvalue H^{-2}, then the equation takes the same form as if derived from the shallow water case, (3):

$$\frac{\partial\phi}{\partial t} + \mathbf{u}_g \cdot \nabla\nabla^2\phi = 0 \tag{28}$$

The difference is because $\mathbf{u}_g.\nabla\phi = 0$ but $\mathbf{u}_g \cdot \nabla\frac{\partial^2\phi}{\partial p^2} \neq 0$. Larichev and McWilliams (1991) and Farge and Sadourny (1991) showed that equation (28) does not exhibit an enstrophy cascade. The solutions remain coherent, and are not sensibly described as homogeneous turbulence of any sort.

Babin et al. (1996,1997) have analysed the behaviour of the inertio-gravity waves in these regimes as well. The overall picture that results is of a flow where on scales $L \geq L_R$, geostrophic and inertio-gravity waves coexist, with weak interaction between them, and there is no systematic transfer of energy or enstrophy either upscale ort downscale in either the potential vorticity or the inertio-gravity waves. However, baroclinic instability will intermittently create disturbances with $L = L_R$. On scales $L \leq L_R$, there is a transfer of rotational energy upscale, towards $L = L_R$, and small gravity wave activity which cascades efficiently to small scales. We can thus expect to find little energy in the regime $L \ll L_R$, except as a result of physical and topographic forcing. The effective L_R for observed flows is difficult to determine, as it depends on the depth scale.

The implication for numerical methods is that an enstrophy dissipation mechanism is required for the rotational flow on scales smaller than L_R, and an energy dissipation mechanism is needed for the divergent flow on scales $\leq L_R$. However, these should not be used where $L > L_R$. It is thus of some importance to determine the effective L_R of the motions being resolved, and that this L_R is properly simulated in the model.

3 NUMERICAL METHODS APPROPRIATE TO STRONGLY ROTATING AND STRATIFIED REGIMES

3.1 General strategy

The basic strategy is to preserve the accuracy of the potential vorticity, and to ensure that inertio-gravity waves and other motions are not excited to an unrealistic degree. Efficiency of operational forecasting requires the use of a long timestep, which has led to the wide use of semi-Lagrangian advection. The idea is therefore to ensure that the model gives a solution with a long timestep which is close to that given by potential vorticity advection appropriate to the asymptotic regime, with other motions damped. If the same model is to be used for smaller scale studies where other motions have to be predicted accurately, a much shorter time step will be needed.

We discuss the regimes of strongly rotating and/or strongly stratified flow analysed above. Operational models need to be good in both cases. Most current operational schemes have been developed from schemes for the shallow water equations, with large equivalent depths (of order 10km). This corresponds to the regime $Fr \ll 1$, and we will show that existing methods are well suited to this case. We then show what further improvements may be needed in other cases.

3.2 Basic semi-implicit method

We consider a simple semi-implicit, semi-Lagrangian scheme. The semi-implicit scheme will be forward weighted in time, and the aim is to show that this damps motions which do not have potential vorticity, leaving the potential vorticity unaffected. This is primarily achieved by using the semi-Lagrangian method to transport the rotational wind components, the potential temperature and the surface pressure. Even if the trajectory calculation is forward weighted in time, the potential vorticity will not be damped. This is because (2) and (4) hold whatever divergent velocity field is used. More time-accuracy for other motions can be obtained by removing the forward weighting. These schemes are discussed in more detail elsewhere in this volume and in the basic review paper of Staniforth and Cote (1991).

The basic scheme for equations (1) is

$$
\begin{aligned}
u^{n+1} &= u_d^n - \delta t\left(\frac{\partial \phi^{n+1}}{\partial x} - f v^{n+1}\right)\\
v^{n+1} &= v_d^n - \delta t\left(\frac{\partial \phi^{n+1}}{\partial y} + f u^{n+1}\right)\\
\theta^{n+1} &= \theta_d^n - \delta t \omega^{n+1}\frac{\partial \hat{\theta}}{\partial p}\\
\frac{\partial \phi^{n+1}}{\partial p} &= \frac{\theta^{n+1}}{\hat{\theta}}\frac{\partial \hat{\phi}}{\partial p}\\
\frac{\partial u^{n+1}}{\partial x} + \frac{\partial v^{n+1}}{\partial y} + \frac{\partial \omega^{n+1}}{\partial p} &= 0\\
p_s^{n+1} &= p_{sd}^n - \delta t\int_0^{p_s}\left(\frac{\partial u^{n+1}}{\partial x} + \frac{\partial v^{n+1}}{\partial y}\right)
\end{aligned}
\tag{29}
$$

The superscript $n+1$ represents time level $t+\delta t$ and n represents time level t. Stability of gravity waves requires that the vertical advection of θ is treated implicitly. To achieve this, and retain time consistency and conservation, requires that the velocity used in the semi-Lagrangian method is at time level $n+1$. Otherwise, different terms which combine to give potential vorticity conservation contain velocities at different time-levels, and conservation will be lost. Greater time accuracy could be obtained if the rotational part of the trajectory was estimated at time level $n+\frac{1}{2}$, and the divergent part at time level $n+1$. Implicit treatment of the trajectory within the semi-Lagrangian advection is not possible, but the implicit correction can be estimated as follows. Assume that the trajectory calculation is applied at time level $n=1$ by using an extrapolation, such as $u^* = 2u^n - u^{n-1}$. Then we define a correction $u' = u^{n+1} - 2u^n + u^{n-1}$, which can be used to generate a correction to the advected fields by using an Eulerian approximation to $\mathbf{u}' \cdot \nabla$ at the arrival point. The same idea can be applied if the scheme is to be a time-centred implicit scheme. Since only u^{n+1} is corrected, the correction can be calculated at the arrival point. Using this method, we obtain

$$\begin{aligned} u' &= \delta t\left(U - \mathbf{u}'\cdot\nabla u - \frac{\partial\phi'}{\partial x} + fv'\right) \\ v' &= \delta t\left(V - \mathbf{u}'\cdot\nabla v - \frac{\partial\phi'}{\partial y} - fu'\right) \\ \theta' &= -\delta t\left(T - \mathbf{u}'\cdot\nabla\theta - \omega'\frac{\partial\hat{\theta}}{\partial p}\right) \\ \frac{\partial\phi'}{\partial p} &= \frac{\theta'}{\hat{\theta}}\frac{\partial\hat{\phi}}{\partial p} \\ \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} &+ \frac{\partial\omega'}{\partial p} = 0 \\ p_s' &= \delta t\left(P - \int_0^{p_s}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right)\right) \end{aligned} \tag{30}$$

U, V, T, P represent known values. The prime superscript has the same meaning for all variables.

We now follow the solution procedure used to estimate ω from (7). Starting with (29), we take the horizontal divergence of the first two equations in (29). This yields

$$\nabla_h^2\frac{\partial\phi^{n+1}}{\partial p} - \frac{\partial^2(fv^{n+1})}{\partial x\partial p} + \frac{\partial(fu^{n+1})}{\partial y\partial p} + O(Ro)\text{ terms} = -\frac{1}{\delta t^2}\frac{\partial^2\omega^{n+1}}{\partial p^2} + X \tag{31}$$

and then substitute in from the remaining equations

$$\delta^2 t\nabla_h^2 N^2\omega^{n+1} + \frac{\partial^2\omega^{n+1}}{\partial p^2} + \delta t^2 f^2\frac{\partial^2\omega^{n+1}}{\partial p^2} + O(Ro)\text{ terms} + \text{ terms in }\nabla f = Y \tag{32}$$

This can be solved for ω^{n+1} with the boundary conditions $\omega^{n+1} = 0$ at $p=0$, $\omega^{n+1} = {\omega^{n+1}}_s$ at $p = p_s$, where

$$(1 + f^2\delta t^2)\omega_s^{n+1} = \frac{C_p\theta_0\kappa}{p_s}\delta t^2\omega_s^{n+1} + \overline{\int_0^p N^2\omega^{n+1}dp} \tag{33}$$

With a large timestep, (32) gives the same approximation to ω^{n+1} as would be derived from (10). Thus time damping in the semi-implicit method will drive the solution towards that of an approximate system obtained by advecting the potential vorticity with a divergent velocity field derived from (10). The analysis in the previous section shows that this will restrict inertia-gravity wave amplitudes to $O(\epsilon)$. If the scale analysis of the previous section applies, the known terms in (32) will have magnitude $\delta t^2 \frac{fU^2}{HL^2}$, and the inertio-gravity wave amplitudes will be restricted to $O(\epsilon)$ unless they are larger in the initial data. If the timestep is greater than $\sqrt(N^2H^2/L^2 + f^2)$, the solution will be a potential vorticity evolution almost free of inertio-gravity waves, which will solve (1) to order Fr^4 or Ro^3 (essentially a nonlinear balance model).

3.3 Numerical accuracy

First compare the above with the more traditional semi-implicit scheme

$$\begin{aligned}
(1 + f^2\delta t^2)u^{n+1} &= u_d^n - \delta t\left(\frac{\partial\phi^{n+1}}{\partial x} - fv^n\right)\\
(1 + f^2\delta t^2)v^{n+1} &= v_d^n - \delta t\left(\frac{\partial\phi^{n+1}}{\partial y} + fu^n\right)\\
\theta^{n+1} &= \theta_d^n - \delta t\omega^{n+1}\frac{\partial\hat{\theta}}{\partial p}\\
\frac{\partial\phi^{n+1}}{\partial p} &= \frac{\theta^{n+1}}{\hat{\theta}}\frac{\partial\hat{\phi}}{\partial p}\\
\frac{\partial u^{n+1}}{\partial x} + \frac{\partial v^{n+1}}{\partial y} + \frac{\partial\omega^{n+1}}{\partial p} &= 0\\
p_s^{n+1} = p_{sd}^n - \delta t\int_0^{p_s}&\left(\frac{\partial u^{n+1}}{\partial x} + \frac{\partial v^{n+1}}{\partial y}\right)
\end{aligned} \tag{34}$$

The semi-Lagrangian advection is now entirely explicit, with an estimate made of the advecting velocity at time $t + \frac{1}{2}\delta t$. The terms $f^2\delta t^2$ represent a simple implicit treatment of the Coriolis terms, necessary for stability. This yields a simpler version of (32):

$$\delta^2 t\nabla_h^2\hat{N}^2\omega^{n+1} + (1 + f^2\delta t^2)\frac{\partial^2\omega^{n+1}}{\partial p^2} = Y \tag{35}$$

This is a good approximation to (32) if the basic state static stability is close to thae actual static stability. Simmons et al. (1978) showed that $\hat{N}^2 > N^2$ is required for the stability of the scheme (34). If this is chosen, but $\hat{N}^2$ is not a good approximation to N^2, (10) shows that the long timestep solution for ω will be underestimated. Since the solution of (10) is the unique ω that maintains large scale balance, the effect will be to overestimate inertio-gravity wave activity. The reduction in ω will also give errors in the potential vorticity advection, though these will not be large as advecetion by the horizontal rotational wind dominates.

The numerical accuracy of this procedure will otherwise depend on how well (32) is solved for ω and how accurately the potential vorticity is advected. The vertical finite difference stencil for (32) is most compact if the Charney-Phillips vertical grid is used (see paper by Davies in this volume and Cullen et al. (1997)), where θ is staggered in the vertical from p, and is at the same level as ω. The Lorenz arrangement will result in extra vertical averaging in the first term of (32). Normally (32) is solved using vertical normal modes, but the structure of the higher vertical modes is substantially different between the Charney-Phillips and Lorenz grids. In particular the phase speeds of the higher modes are much lower on the Lorenz grid, which reduces the effective N^2 and hence will exaggerate the value of ω^{n+1}. The horizontal approximation to (32) is optimal in a spectral model, where vorticity, divergence and potential temperature have the same spectral representation. It would also be optimal in a finite difference model where potential temperature, vorticity and divergence are held at the same points in the horizontal. Some loss of accuracy is unavoidable if velocity components are used as variables, because then the vorticity and divergence are either calculated at different points (e.g. Arakawa C grid), or can only be calculated by averaging (e.g. Arakawa B grid). On the C grid, the term $\nabla_h^2 N^2 \omega^{n+1}$ is not averaged in the horizontal but the term $f^2 \frac{\partial^2 \omega^{n+1}}{\partial p^2}$ is averaged over 9 points. On the B grid, the first term is averaged over 9 points and the second over 4 points. In the case $L < L_R$, the first term dominates and so averaging is less damaging in the second term. The C grid is thus preferred. The converse applies if $L > L_R$. Studies, e.g. Bryan (1989), confirm that the C grid is preferable to the B grid on horizontal scales less than L_R, and vice versa.

The semi-Lagrangian advection should ensure accurate transport of the vertical component of vorticity, the potential temperature, and the surface pressure. Conservation of a linearised 'quasi-geostrophic' approximation to the potential vorticity like (25)

$$f\frac{N^2p}{R\Pi} + \frac{N^2p}{R\Pi}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + f\frac{\partial \theta}{\partial p} \tag{36}$$

with barotropic part

$$f/p_0(1-(p_s-p_0)/p_0) + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)/p_0$$

will be assured if a consistent divergence χ is used to update $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$, to calculate the term $\omega\frac{\partial \hat{\theta}}{\partial p}$ in the θ equation, and to update p_s. Nonlinear conservation of the exact potential vorticity (2) requires that the semi-Lagrangian method is used to treat vortex stretching (which appears as a term $(\zeta + f)\chi$ in the equation for the vertical component of vorticity) and the vertical advection of the total θ including the basic state $\hat{\theta}$. The former has been used in idealised vortex calculations, e.g. Chorin (1986), but not in meteorology. If, in addition, the trajectory calculation is volume preserving, as in Scroggs and Semazzi (1995), then accurate conservation should be achieved.

The scheme (29) or (30) can be written in the form (15). It thus contains a proper representation of the effects discussed in section 2.3, provided the **Q** matrix is properly included in the semi-implicit scheme. This is achieved if (29) is used to approximate the dynamics, but requires implicit treatment of appropriate phyiscs increments as shown in (21). Thus we can write

$$\mathbf{Q}\begin{pmatrix} u' \\ v' \\ \omega' \end{pmatrix} + \frac{1}{\delta t}\begin{pmatrix} fv' - c_D u' \\ -fu' - c_D v' \\ \frac{1}{\hat{\theta}}\frac{\partial \hat{\phi}}{\partial p}\theta' \end{pmatrix} = \mathbf{H}' \tag{37}$$

where

$$\mathbf{H}' = \begin{pmatrix} -f\frac{\partial \phi'}{\partial y} + c_D\frac{\partial \phi'}{\partial x} \\ f\frac{\partial \phi'}{\partial x} + c_D\frac{\partial \phi'}{\partial y} \\ S_h \end{pmatrix} + \text{known values} \tag{38}$$

Implementation is best achieved by constructing an implicit equation for ϕ from a rearrangement of (37) and (38):

$$\mathbf{Q} + \frac{1}{\delta t}\begin{pmatrix} -c_D & f & 0 \\ -f & -c_D & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u' \\ v' \\ \omega' \end{pmatrix} + \begin{pmatrix} f\frac{\partial \phi'}{\partial y} - c_D\frac{\partial \phi'}{\partial x} \\ -f\frac{\partial \phi'}{\partial x} - c_D\frac{\partial \phi'}{\partial y} \\ \frac{1}{\delta t}\frac{\partial \phi'}{\partial p} \end{pmatrix} = \mathbf{H} \tag{39}$$

where **H** is now known. $\mathbf{u}'$ can be eliminated from (39) using the continuity equation, allowing a solution for ϕ', and hence the other unknown variables. The optimum grid design for solving (39) is the same as that for the purely dynamical scheme (32).

4 SUMMARY

The main conclusions from a study of the behaviour of the exact solutions for strongly rotating and/or stratified flow are:

i) Current semi-implicit, semi-Lagrangian numerical procedures are well founded for strongly stratified flow, but need to be extended to include implicit calculation of the trajectory to give accurate treatment of strongly rotating flow.

ii) The concept of two-dimensional turbulence is only applicable for horizontal scales smaller than the Rossby radius of deformation. Efficient parametrization of the inertio-gravity wave energy cascade is needed on scales $\leq L_R$, but not on larger scales.

iii) The Charney-Phillips vertical staggering is optimal for the dynamics in either regime. There are advantages in treating rotational and divergent parts of the horizontal wind as separate variables.

iv) There is scope for modification of the semi-Lagrangian advection procedures to improve potential vorticity conservation. This can be achieved without using potential vorticity as a variable, which would be difficult in an operational model.

v) Sub-grid models must be designed so that the analytic solution of the explicit equation plus sub-grid model stays smooth on the filtering scale chosen.

v) The physics should be more closely coupled to the dynamics by including linearised corrections to appropriate schemes in the implicit step in the dynamics. In particular, it may be worth treating convection this way.

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