TECHNICAL MEMORANDUM

115

Energy conserving Galerkin finite element schemes

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ABSTRACT

Energy conserving finite element schemes are given for the divergent equations of meteorological flow. The schemes are formulated for the divergent shallow water equations. One of the schemes allows linear finite element spaces for all fields.

CONTENTS

1. INTRODUCTION

For extended time integrations of the equations of atmospheric motion the importance of observing energy and other conservation properties of the finite difference equations has been recognised. The schemes in use are based on ideas developed by Arakawa (1966). The use of energy conserving schemes generally improves the nonlinear stability of the schemes and therefore reduces the need for numerical diffusion. For climate models, conservation properties of numerical approximations may be a desirable property in itself.

For the purposes of medium range weather prediction, it may be desirable to refine the grid in certain areas of particular interest for the forecast in the target area. Such models with grid refinement may suffer from nonlinear instability. Consequently short range forecast models of this type are usually implemented with relaxation procedures, as proposed by Davies (1976). Since conservation properties of schemes are known to enhance the nonlinear stability, it is interesting to formulate conserving schemes for irregular grids.

Finite element schemes can be supposed to have a good nonlinear stability because of the reduced aliasing error of the Galerkin method. Furthermore, they automatically transfer their conservation properties to the case of irregular resolution. Standard finite element schemes can be expected to conserve quadratic moments, from which energy conservation for the non-divergent flow follows (see Fix (1975) and Jespersen (1974)). Also, the semi-discretized equations, obtained by approximating only the vertical coordinate in σ-coordinate equations and considering the horizontal coordinates as continuous, are normally energy conserving (see Burridge et al. (1985)). Again in this case the energy is a second order moment.

Conserving schemes, including those with energy conservation were formulated by Cliffe (1981) for the Boussinesq equations. A relation between the approximation spaces for pressure and temperature is necessary to obtain energy conservation. In particular, linear elements for the velocities and pressure are not admitted by Cliffe (1981) for an energy conserving scheme.

This paper presents some energy conserving finite element schemes for the primitive meteorological equations. Schemes are formulated for the two-dimensional shallow water equations. All schemes presented can easily be generalised to energy conserving schemes of the σ-system equations. This may be achieved by starting from the semi-discretized equations given by Burridge et al., 1986. One of the schemes requires, as in Cliffe (1981), a relation between the approximation spaces for pressure and temperature. Another scheme uses only linear elements for all fields, and therefore may be a good candidate for implementation in large models.

Since the schemes use only general relations between the basis function spaces, the developments are valid for other Galerkin schemes, such as the spectral method or the parameter fluid models proposed by Steppeler (1979a, b). Questions of practical implementation are, however, discussed only for the use of linear finite elements for the velocity field.

2. GALERKIN PROJECTIONS

For fields ϕ, we assume the following representation:

$$\tilde{\phi}\ (\underline{r}) = \sum_{\nu=1}^{N} \phi_\nu\ e_\nu(\underline{r}) \tag{1}$$

In equ.(1), $\underline{r}$ is the vector x,y, with x,y being horizontal coordinates. Depending on the choice of the basis functions e_ν, different approximating spaces $\Omega_1, \Omega_2 \ldots$ are obtained. We consider here only linear and quadratic finite element (FE) spaces, though this specialisation is used only to discuss questions of practical implementation.

To formulate FE schemes for prognostic equations, it is necessary to approximate a rather general field $\phi(\underline{v})$ by a function $\breve{\phi}(\underline{v})$ of the space given by equ.(1). This is achieved by the Galerkin projection, where the coefficients ϕ_ν in equ.(1) are defined by:

$$(e_\xi,\breve{\phi}) = (e_\xi,\phi), \quad \xi\varepsilon\{1,\ldots,N\} \tag{2}$$

The scalar product (a,b) is defined as:

$$(a,b) = \int a(\underline{r})\, b(\underline{r}) d\mu \tag{3}$$

Here, it is sufficient to assume the measure $d\mu$ to be of the form

$$d\mu = w(\underline{r}) dxdy, \tag{4}$$

with w being a positive continuous function. The relation between the fields ϕ and $\breve{\phi}$, given by equ.(2), can be written as:

$$\breve{\phi} = G\phi \tag{5}$$

with G being the Galerkin projection operator.

The following properties of G can easily be verified:

$$\begin{aligned} &G^2 = G.G = G \\ &(Ga,b) = (a,Gb) \\ &G(a\phi_2+b\phi_2) = aG\phi_1+bG\phi_2 \end{aligned} \tag{6}$$

By choosing different spaces of basis functions, and different w in equ.(4), different Galerkin projections G_1, G_2,..., belonging to scalar products $(\ ,\ )_1$ and $(\ ,\ )_2$ ..., can be defined.

For prognostic equations of the form

$$\dot{\phi} = RS, \tag{7}$$

the approximation

$$\dot{\phi} = GRS \tag{8}$$

is referred to as the standard FE-scheme. Of course, in equ.(8), a different G can be used for different components of ϕ.

3. ENERGY CONSERVING SCHEMES FOR THE SHALLOW WATER EQUATIONS

The schemes are presented here for the x-y plane. The generaliation to the rotating sphere is obvious. An energy conserving scheme for the shallow water equations is

$$\begin{aligned} \dot{u} &= -G_1(u\ u_x + v\ u_y + H_x) \\ \dot{v} &= -G_1(u\ v_x + v\ v_y + H_y) \\ \dot{H} &= -G_2((u\ H)_x + (u\ H)_y) \end{aligned} \tag{9}$$

The above scheme will be referred to as scheme A. Let Ω_1 and Ω_2 be the spaces belonging to the projections G_1, G_2 and $(\ ,\ )_1$ and $(\ ,\ )_2$ be the corresponding scalar products. The conditions of energy conservation are:

$$\Omega_2 \subset \Omega_1^2 \tag{10}$$

$$(a,b)_1 = \int H(r)a(\underline{r})b(\underline{r})dxdy \tag{11}$$

$$(a,b)_2 = \int a(r)b(r)dxdy \tag{12}$$

The quadratic space ${\Omega^2}_1$ of Ω_1 is defined as the space generated according to equ.(1) by all products $e_\nu(\underline{r})e_\mu(\underline{r})$, where the e_ν form a basis of Ω_1.

In one space dimension, the quadratic space of the space of linear splines is the space of quadratic splines with the same node points. In two space dimensions, not every quadratic space can be used. For example, the quadratic element space defined by Steppeler (1976) is not the quadratic space of the corresponding linear element scheme. For a regular rectangular mesh the grid for the definition of the quadratic space of the linear splines is given in Fig. 1.

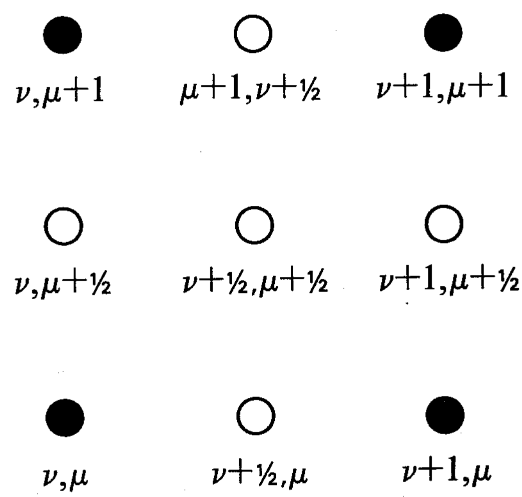

Fig. 1 The grid for the definition of the quadratic space of linear splines for regular rectangular resolution.

Let $e_\nu(x)$ be the basis functions of linear splines in one space dimension, and let the basis functions of linear splines in two space dimensions be given as:

$$b^1_{\nu,\mu}(x,y) = e_\nu(x)e_\mu(y) \tag{13}$$

Then the following basis functions are associated with the node points in Fig. 1 for its quadratic space:

$$b^2_{\nu,\mu} = b^1_{\nu,\mu} \tag{14}$$

$$b^2_{\nu+\frac{1}{2},\mu} = e_\nu(x)e_{\nu+1}(x)e_\mu(y) \tag{15}$$

$$b^2_{\nu,\mu+\frac{1}{2}} = e_\nu(x)e_\mu(y)e_{\mu+1}(y) \tag{16}$$

$$b^2_{\nu+\frac{1}{2},\mu+\frac{1}{2}}(x,y)=e_\nu(x)e_{\nu+1}(x)e_\mu(y)e_{\mu+1}(y) \tag{17}$$

The basis function $b^2_{\nu+\frac{1}{2},\mu+\frac{1}{2}}$ was missed out in the quadratic spline space defined by Steppeler (1976). The condition given by equ.(10) is rather similar to a condition used by Cliffe (1981) to obtain energy conservation. In this study the observation has already been made that not every space of quadratic splines for the pressure field leads to energy conservation.

To prove the energy conservation of equs.(9), form the time derivative of the total energy and denote the vector u,v by $\underline{u}$. We assume that the FE discretization of $\underline{u}$ implies $\underline{u}=0$ at the boundaries of the region.

$$\begin{aligned}\dot{E} &= \frac{\partial}{\partial t} \int \tfrac{1}{2}\, H(\underline{u}^2 + H)\,dxdy \\ &= \tfrac{1}{2}\,(\dot{H},\underline{u}^2)_2 + (\dot{H},H)_2 + (\underline{u},\dot{\underline{u}})_1\end{aligned} \tag{18}$$

The definitions equs.(11) and (12) have been used in equ.(18). According to equs.(9) we obtain:

$$\begin{aligned}\dot{E} &= -(G_2\ \mathrm{div}(H\underline{u}),\tfrac{1}{2}\underline{u}^2+H)_2 \\ &\quad -(\underline{u},G_1\mathrm{grad}(\tfrac{1}{2}\underline{u}^2+H))_1\end{aligned} \tag{19}$$

Using equs.(6), one can eliminate the Galerkin projections in equ.(19):

$$\dot{E} = -(\mathrm{div}\ H\underline{u}, \tfrac{1}{2}\underline{u}^2+H)_2$$
$$-(uH, (\tfrac{1}{2}\underline{u}^2+H)_x)_2 - (vH, (\tfrac{1}{2}\underline{u}^2+H)_y)_2 \qquad (20)$$

Performing a partial integration in equ.(20), using the condition that $\underline{u}$ is 0 at the boundaries, one can see that the energy is conserved.

For easy practical implementation it may be convenient to have a scheme which allows linear finite element spaces for the approximation of all fields. Such schemes can be obtained using a form of the shallow water equations used in Steppeler (1976) to obtain difference schemes with conservation properties. It will be referred to as scheme B.

$$\begin{aligned} \dot{u} &= G_1(\tilde{\eta}\ v - (G_2(\tfrac{1}{2}\underline{u}^2 + H)_x) \\ \dot{v} &= G_1(-\tilde{\eta}\ u - (G_2(\tfrac{1}{2}\underline{u}^2 + H)_y) \\ \dot{H} &= -G_2 \mathrm{div}(H\ \underline{u}) \\ \eta &= v_x - u_y \end{aligned} \qquad (21)$$

Again we require that G_1 corresponds to the scalar product given in equ.(11) and G_2 corresponds to that of equ.(12). $\tilde{\eta}$ can be interpolated arbitrarily from η. No condition on the approximation spaces belonging to G_1 and G_2 is imposed.

To prove energy conservation for equs.(21), we consider the energy equation:

$$\begin{aligned} \dot{E} &= \frac{\partial}{\partial t} \int H.\tfrac{1}{2}(\underline{u}^2+H)\,dxdy \\ &= -(\underline{u}, G_1(\mathrm{grad}\ G_2(\tfrac{1}{2}\underline{u}^2+H)))_1 - (G_2 \mathrm{div}\underline{u}H, \tfrac{1}{2}\underline{u}^2+H)_2 \end{aligned} \qquad (21)$$

Using equs.(6) and performing a partial integration, observing the boundary conditions for $\underline{u}$, we obtain:

$$\dot{E} = -(\mathrm{div}\ H\underline{u},\ G_2(\tfrac{1}{2}\underline{u}^2+H))_2 + (G_2 \mathrm{div}\ \underline{u}H,\ \tfrac{1}{2}\underline{u}^2+H)_2 \qquad (22)$$

Using again equ.(6), we see that the right hand side of equ.(22) is zero.

4. A NUMERICAL CALCULATION

The possible impact of energy conservation on the stability of meteorological models has to be explored by two dimensional models. Here only the result of one dimensional calculations is given. Since a very small timestep was used, time discretization does not do much to destroy energy conservation.

A one dimensional model was obtained by putting v and all y-derivatives to 0 in equ.(9) and equ.(21). The equations can be nondimensionalized using a space scale X_o and a time scale t_o. The finite element scheme was obtained by using a periodic channel with 22 node points being X_o apart, except for points 10 to 12, which were only $\frac{1}{2}X_o$ apart. A gravity wave was obtained by using as initial condition u=0 and H=1 for $x \in (5X_o, 10X_o)$ and H=.8 for $x \notin (5X_o, 10X_o)$.

For comparison, a nonconserving scheme, referred to as the control scheme, was obtained by replacing G_1 in eq.(9) by G_2. The timestep used was $.01t_o$.

Figs. 2-4 give the energy diagram for $t \in (0, 10t_o)$ for the control scheme and schemes A and B. A better conservation of energy by schemes A and B is apparent. In this simple example no nonlinear instability occurs, even for very long integrations, for any scheme.

5. CONCLUSIONS

Two energy conserving Galerkin finite element schemes were obtained for the primitive meteorological equations. One of them, the B-scheme, allows the use of linear elements for all fields, and is therefore relatively easy to implement in large models.

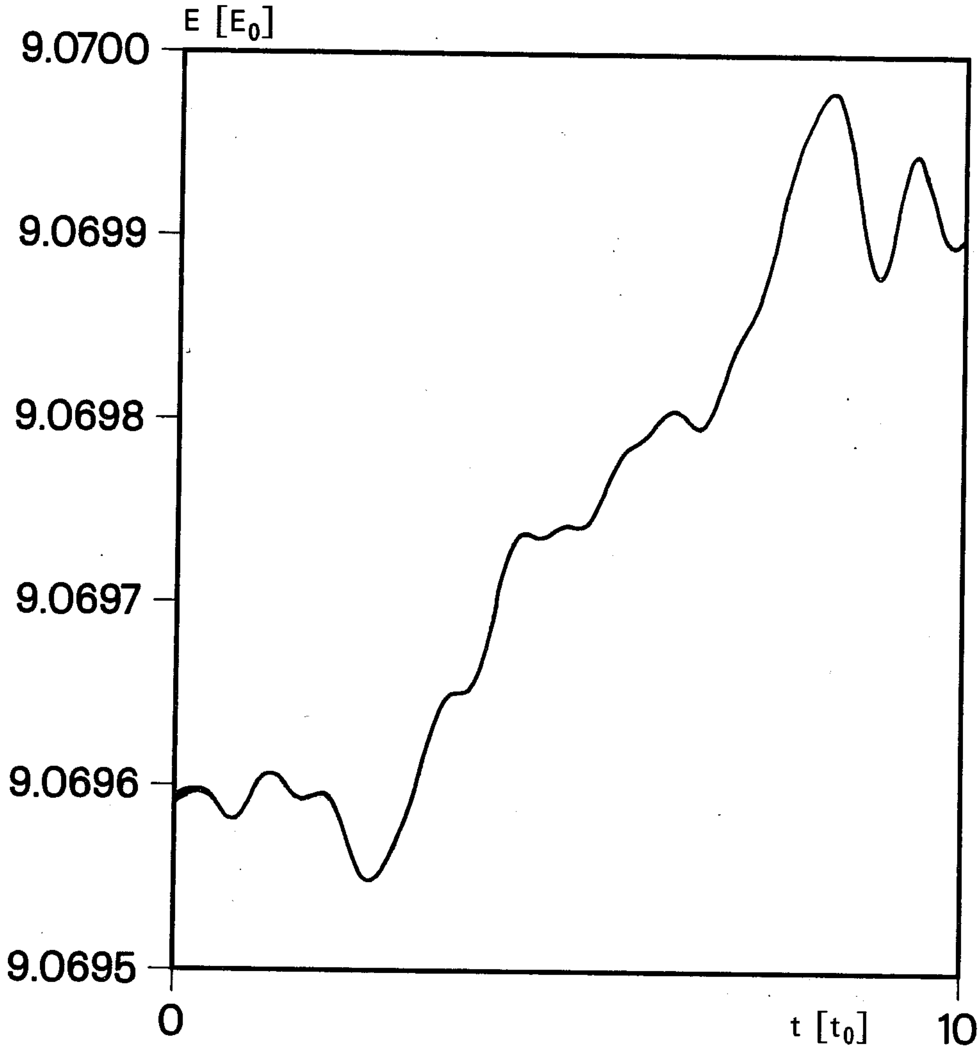

Fig. 2 Energy diagram with the (nonconserving) control finite element scheme for a one dimensional gravity wave.

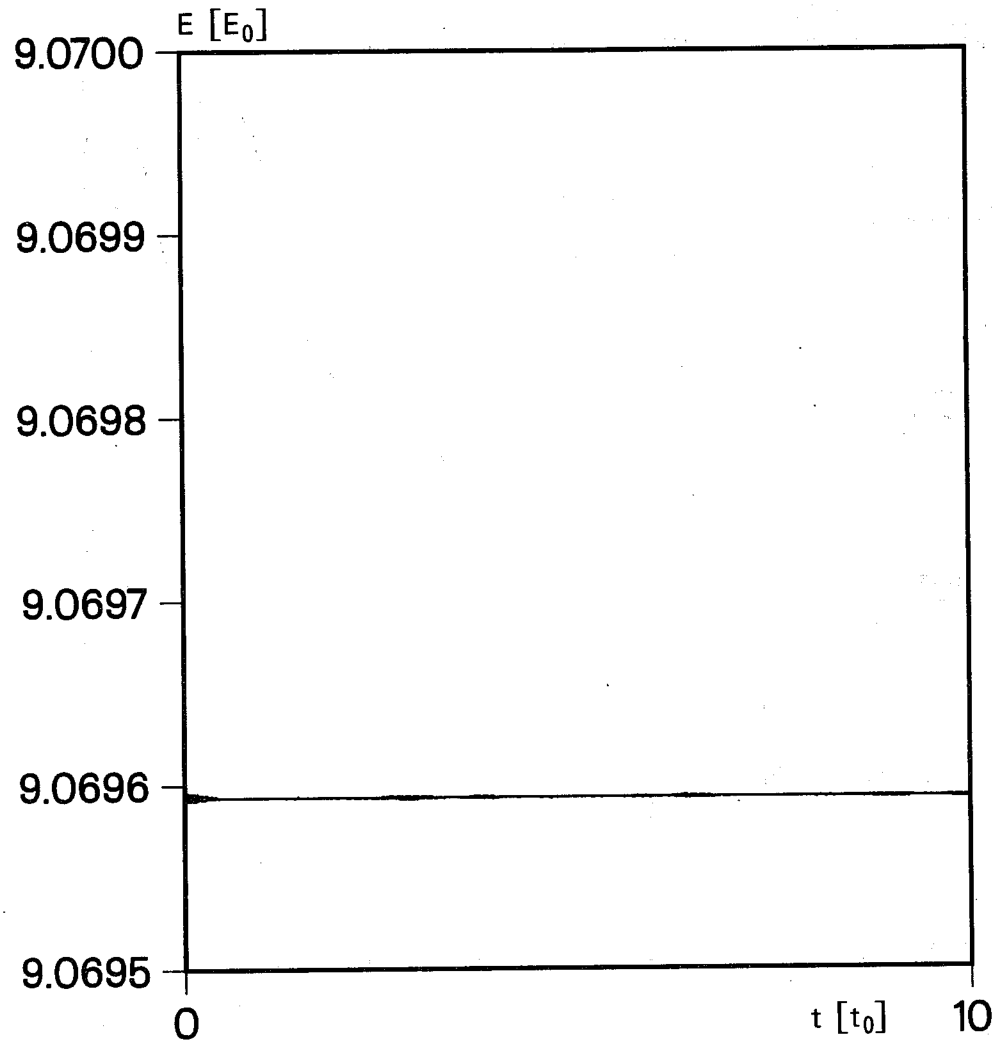

Fig. 3 As Fig. 2, for scheme A.

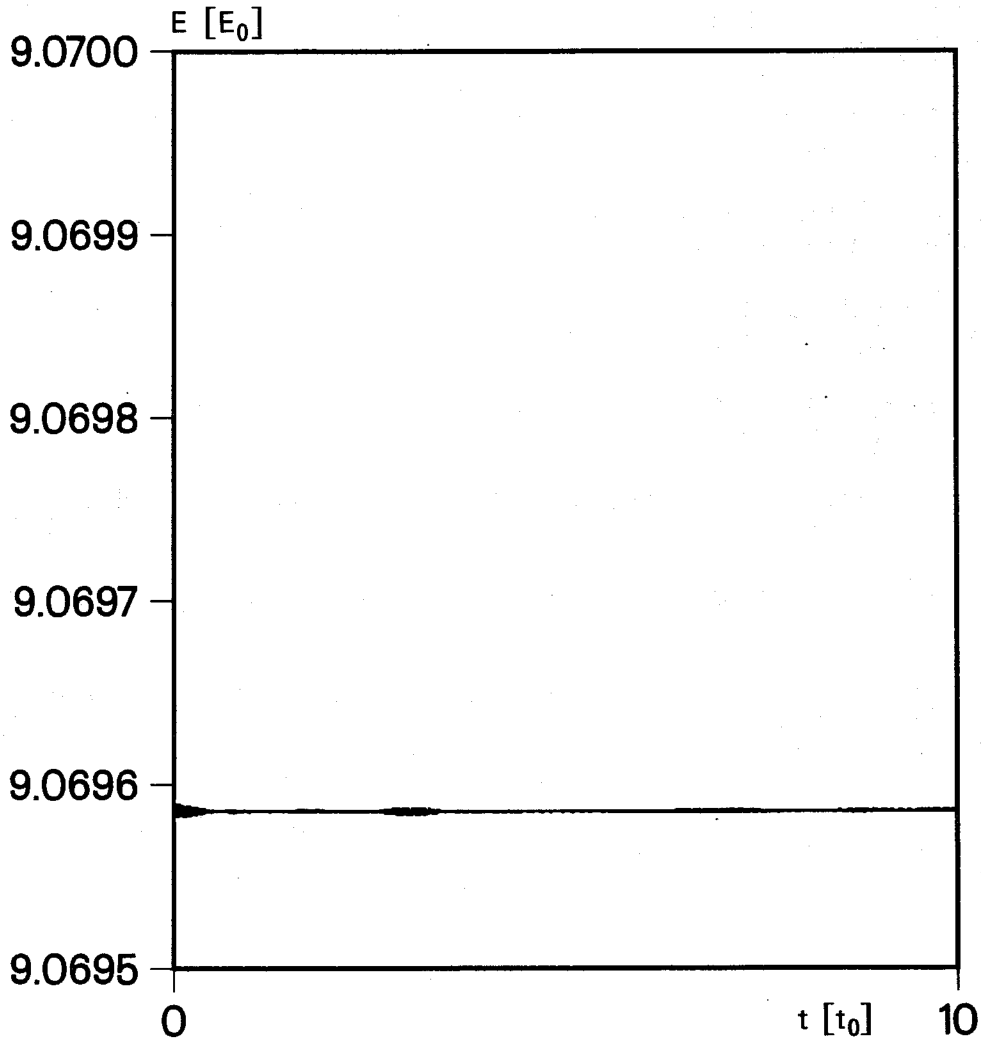

Fig. 4 As Fig. 2, for scheme B.

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