CONVERGENCE OF ASSIMILATION PROCEDURES

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These notes deal with a number of mathematical problems related to four-dimensional data assimilation. They are based on a general criterion for convergence of an assimilation procedure, which is derived in Sections 1 and 2, and then applied to various cases.

The basis for the content of these notes has been described in detail in a previous publication (Talagrand, 1977) which will be hereafter referred to as T77.

1. MATHEMATICAL PRELIMINARIES

We will formalise the assimilation problem by assuming that we observe the time evolution of a physical system (which will in applications be the atmosphere, but no specific hypothesis is necessary at this stage). The system will be assumed to be made of two parts: a first part X (e.g. the mass field) which is observed at successive instants, and whose state at any time is defined by the values of a finite number p of independent parameters; and a second part Y (e.g. the wind field) which is to be reconstituted through an assimilation process, and whose state at any time is defined by the values of q independent parameters.

The time evolution of the system is supposed to be described by a set of $r = p+q$ differential equations of first order with respect to time, which can be summed up in the two vector equations

$$\frac{dX}{dt} = F[X,Y] \tag{1a}$$

$$\frac{dY}{dt} = G[X,Y] \tag{1b}$$

where F and G are respectively a p-valued and a q-valued function of X and Y. F and G could be assumed to depend also on time, without this resulting in any modification to the sequel. Equations (1) can be thought of as being, for instance, the meteorological primitive equations.

An arbitrary initial time t_o being chosen, any initial conditions $[X_o, Y_o]$ at t_o define a unique solution to Equations (1). The values assumed by that solution at time t will be noted $[X(X_o,Y_o;t), Y(X_o,Y_o;t)]$. Any initial perturbations $[\Delta X_o, \Delta Y_o]$ on $[X_o,Y_o]$ define a solution

$$[X(X_o,Y_o;t) + \Delta X(t), Y(X_o,Y_o;t) + \Delta Y(t)] ,$$

where the time evolution of ΔX and ΔY is described by the perturbation system in the vicinity of the solution $[X(X_o,Y_o;t),\ Y(X_o,Y_o;t)]$

$$\frac{d\Delta X}{dt} = F[X(X_o,Y_o;t) + \Delta X,\ Y(X_o,Y_o;t) + \Delta Y] - F[X(X_o,Y_o;t), Y(X_o,Y_o;t)] \quad (2a)$$

$$\frac{d\Delta Y}{dt} = G[X(X_o,Y_o;t) + \Delta X,\ Y(X_o,Y_o;t) + \Delta Y] - G[X(X_o,Y_o;t), Y(X_o,Y_o;t)] \quad (2b)$$

These equations are readily obtained from (1). Linearising them with respect to ΔX and ΔY (and changing ΔX and ΔY to δX and δY in order to avoid any confusion with (3)) leads to

$$\frac{d\delta X}{dt} = \frac{DF}{DX}(t)\ \delta X + \frac{DF}{DY}(t)\ \delta Y \quad (3a)$$

$$\frac{d\delta Y}{dt} = \frac{DG}{DX}(t)\ \delta X + \frac{DG}{DY}(t)\ \delta Y \quad (3b)$$

where $\frac{DF}{DX}(t)$ is the p x p jacobian matrix made up of the partial derivatives of F with respect to the components of X. The argument t means that these derivatives are taken at point $[X(X_o,Y_o;t),\ Y(X_o,Y_o;t)]$. $\frac{DF}{DY}(t)$, $\frac{DG}{DX}(t)$, $\frac{DG}{DY}(t)$ are similarly defined jacobian matrices, with respective dimensions p x q, q x p and q x q. System (3) is the <u>linearised perturbation system</u> in the vicinity of the solution $[X(X_o,Y_o;t),\ Y(X_o,Y_o;t)]$. Unless the basic equations (1) are themselves linear, there is one different such system for every solution of (1).

The solution at any time t of system (3) depends linearly on the initial conditions $[\delta X_o, \delta Y_o]$ at time t_o. This is expressed by the following equations

$$\delta X(t) = R_x^x(t,t_o)\ \delta X_o + R_x^y(t,t_o)\ \delta Y_o \quad (4a)$$

$$\delta Y(t) = R_y^x(t,t_o)\ \delta X_o + R_y^y(t,t_o)\ \delta Y_o \quad (4b)$$

where $R_x^x(t,t_o)$, $R_x^y(t,t_o)$, $R_y^x(t,t_o)$, $R_y^y(t,t_o)$ are four matrices with respective dimensions p x p, p x q, q x p, q x q, which together make up a square matrix $R(t,t_o)$ of order r

$$R(t,t_o) = \begin{pmatrix} R_x^x\ (t,t_o) & R_x^y\ (t,t_o) \\ & \\ R_y^x\ (t,t_o) & R_y^y\ (t,t_o) \end{pmatrix} \tag{5}$$

$R(t,t_o)$ is called the <u>resolvent matrix</u> of the linearised system (3) between times t_o and t.

We will be interested in the linearised perturbation system (3) for the following reason: for "small" initial perturbations $[\Delta X_o, \Delta Y_o]$, the time evolution of the resulting perturbation $[\Delta X(t), \Delta Y(t)]$ is described to some approximation by system (3). This vague statement is made precise by the following theorem (for a proof, see e.g. Coddington and Levinson (1955)).

<u>Theorem (p) Given initial perturbations $[\Delta X_o, \Delta Y_o]$ at time t_o, the difference at any given time t between the corresponding solutions of the exact and linearised perturbation systems (2) and (3) is an infinitesimal of higher order than $[\Delta X_o, \Delta Y_o]$.</u>

This can be expressed by the following equations

$$\Delta X(t) = R_x^x\ (t,t_o)\Delta X_o + R_x^y\ (t,t_o)\Delta Y_o + o\ (\Delta X_o, \Delta Y_o) \tag{6a}$$

$$\Delta Y(t) = R_y^x\ (t,t_o)\Delta X_o + R_y^y\ (t,t_o)\Delta Y_o + o\ (\Delta X_o, \Delta Y_o) \tag{6b}$$

where, following a standard notation, $o(\Delta X_o, \Delta Y_o)$ denotes an infinitesimal of higher order than $[\Delta X_o, \Delta Y_o]$, i.e. a function of $[\Delta X_o, \Delta Y_o]$ such that the ratio $\dfrac{|o(\Delta X_o, \Delta Y_o)|}{|\Delta X_o| + |\Delta Y_o|}$ tends to 0 when ΔX_o and ΔY_o tend to 0.

Theorem (p) essentially means that the values $[X(X_o,Y_o;t),\ Y(X_o,Y_o;t)]$ assumed at time t by a solution of the basic system (1) are continuous and differentiable functions of the initial conditions $[X_o,Y_o]$, and that the corresponding partial derivatives are the entries of the resolvent matrix $R(t,t_o)$ of (5).

Some further properties of the resolvent matrix will be useful. Since the choice of the initial time t_o is arbitrary, $R(t',t'')$ is defined for any two times t' and t" at which the solution $[X(X_o,Y_o;t),\ Y(X_o,Y_o;t)]$ is itself defined, whether $t' < t''$ or $t'' \geqslant t'$. Since integrating system (3) from t' to t", and then to a third time t''' will produce the same result as integrating directly from t' to t''', the corresponding resolvent matrices must satisfy the following relationship

$$R(t''', t'') \, R(t'',t') = R(t''',t') \qquad (7)$$

which, when t''' = t', reduces to

$$R(t',t'') \, R(t'',t') = I \qquad (8)$$

where I is the unit matrix of order r.

Taking into account decomposition (5), each of the two above relationships can be readily transformed into four relationships between the corresponding matrices R_x^x, R_x^y.....

2. CONVERGENCE OF AN ASSIMILATION PROCEDURE

For the sake of definiteness, we will make the following two hypotheses about the assimilation process

a) The part X has been observed at N successive instants $t_1, t_2, \ldots, t_N$ and the assimilation is performed according to the simplest forward-backward procedure: the numerical model is integrated alternatively forward and backward in time over the time period $[t_1, t_N]$. Whenever the model time reaches an observation time t_i, whether in a forward or in a backward integration, the values predicted for X are replaced with the observed values, while the values predicted for Y are not modified. The model integration is then resumed, and carried to the next introduction time, at which a new updating is performed.

This procedure calls for an important remark. The nature of the parameters which make up X is imposed by the observations, but the nature of the parameters which make up Y is not imposed by the conditions of the problem. The nature of Y can be chosen arbitrarily under the only condition that X and Y together uniquely define the complete state of the system. For instance, in the case of an atmospheric model, with X representing, say, the mass field, one can choose for Y either the velocity field $\mathbb{V}$, or the momentum field $\rho\mathbb{V}$, or any other combination of the mass and velocity field which, together with the mass field, completely defines the state of the flow. These different choices are obviously not equivalent for the assimilation. The nature of X being given, there are infinitely many possible choices for the nature of Y. For this reason, the hypothesis that Y is not modified at an introduction time is much less restrictive than could *a priori* seem. We will come back later to this point and will assume for the time being that one particular arbitrary choice has been made for Y.

b) Both the assimilating model and the observations are perfect. This means that the observations $X(t_1), \ldots, X(t_N)$ are exactly compatible with one solution of the model equations. This solution will be called the observed solution. The hypothesis made here is the classical "identical twin" hypothesis. We will see later how the results to be derived below are modified when this hypothesis is relaxed.

2.1 The amplification matrix over an assimilation cycle

We shall choose the latest observation time t_N as the arbitrary origin of the successive forward-backward assimilation cycles. Setting $t_N - t_1 = T$, it will be convenient to introduce along each assimilation cycle an auxiliary time variable τ, defined modulo 2T. This variable will increase from 0 to T in the backward phase of the cycle, and from T to 2T in the forward phase. To each of its values τ, there corresponds a unique value of the real time. To any value t of the real time, $t_1 < t < t_N$, there correspond two values τ and τ' of the auxiliary time, such that $\tau + \tau' = 2T$, and belonging respectively to the backward and forward phases of the assimilation cycle. The instants in the cycle when observations of X are introduced into the model will be denoted τ_1 (coinciding with t_N), $\tau_2, \ldots, \tau_N$ (coinciding with t_1), $\tau_{N+1}, \ldots, \tau_{2(N-1)}$ (coinciding with t_{N-1}). We shall define $M = 2(N-1)$. M introductions of observations are performed in the course of one assimilation cycle.

For any two values τ and τ' of the auxiliary time variable, it will be convenient to denote by $R(\tau', \tau)$ the resolvent matrix (5) between the corresponding values of the real time. A similar notation will be used for the submatrices R_X^X, R_X^Y, ...

Let us now consider the difference $(\Delta X, \Delta Y)$ between the assimilating model and the observed solution. Between two observation times, this difference varies according to the perturbation system (2) where the "unperturbed" solution $[X(X_o, Y_o; t), Y(X_o, Y_o; t)]$ is now the observed solution $[X(t), Y(t)]$. At an observation time τ_i, ΔX is set equal to 0, while ΔY remains unchanged. Let ΔY_n be the Y-difference at the end of the n-th assimilation cycle. In the (n+1)-st cycle, starting at time τ_1, the model integration from τ_1 to τ_2 will produce a difference ΔX which will be then set equal to 0, and a difference $\Delta Y(\tau_2)$ which, according to (6b) is equal to

$$\Delta Y(\tau_2) = R_y^y(\tau_2,\tau_1)\ \Delta Y_n + o(\Delta Y_n)$$

Similarly, the subsequent integration of the model from τ_2 to τ_3 will produce a difference

$$\Delta Y(\tau_3) = R_y^y(\tau_3,\tau_2)\ \Delta Y(\tau_2) + o(\Delta Y(\tau_2))$$

$$= R_y^y(\tau_3,\tau_2)\ R_y^y(\tau_2,\tau_1)\ \Delta Y_n + o(\Delta Y_n)$$

This argument, carried out over the complete assimilation cycle, shows that the difference ΔY_{n+1} at the end of the (n+1)-st cycle will be

$$\Delta Y_{n+1} = A\ \Delta Y_n + o(\Delta Y_n) \tag{9}$$

where A is the matrix

$$A = R_y^y(\tau_1,\tau_M)\ R_y^y(\tau_M,\tau_{M-1}) \ldots\ldots\ R_y^y(\tau_2,\tau_1) \tag{10}$$

A is the product of M square matrices of order q each of which represents the effect of the assimilation over one interval (τ_i, τ_{i+1}). A will be called the amplification matrix of the difference ΔY over one assimilation cycle. It is entirely determined by the linearised perturbation system (3) in the vicinity of the observation solution and, more precisely, by only one, namely R_y^y, of the four submatrices which make up R (see Eq. (5)). It must be noted that, contrary to R, R_y^y does not satisfy a "contracting" relationship of type (7), so that expression (10) cannot be written in a more concise form.

Now, according to a general result of matrix algebra, the relevant parameter for the behaviour of ΔY_n as n tends to infinity is the spectral radius $\rho(A)$, which is by definition the largest modulus of the eigenvalues of A (see e.g. Varga (1962):

- if $\rho(A)$ is strictly less than 1, ΔY_n will tend to 0 as n increases to infinity, provided the initial difference ΔY_o is small enough so that the $o(\Delta Y_o)$ term in (9) is negligible compared with $A\Delta Y_o$.

- if $\rho(A)$ is larger than 1, the components of ΔY_n along eigenvectors corresponding to eigenvalue(s) with modulus larger than 1, will be amplified by the assimilation, and ΔY_n will not tend to 0.

- finally, if $\rho(A)$ is exactly 1, the component of ΔY_n along eigenvectors corresponding to eigenvalue(s) with modulus equal to 1, will be neither amplified nor reduced in the product $A\Delta Y_n$, and the behaviour of ΔY_n as n tends to infinity will depend on the higher order term $o(\Delta Y_n)$ in (9).

We see that $\rho(A) \leqslant 1$ is a necessary condition for convergence of an assimilation, and $\rho(A) < 1$ a sufficient condition, when convergence is defined as follows: there exists some number $\varepsilon > 0$ such that ΔY_n will tend to 0 as n tends to infinity provided the initial difference ΔY_o is less than ε.

For the sake of simplicity, we have assumed that the "same" p parameters, making up the vector X, are observed at the successive times $t_1, t_2, \ldots, t_N$. This assumption is in effect not necessary and an amplification matrix of type (10) can be obtained through a similar derivation when the nature, or even the number of the observed parameters varies with the observation time t_i.

Also, we have considered only the case of a forward-backward assimilation which, being an exactly iterative process lends itself more easily to a rigorous mathematical treatment. At the price of a somewhat more complicated mathematical formalism, the approach presented above can be extended to a purely forward assimilation, in which the model is always integrated forward in time, and new data are constantly introduced. In this case too, the convergence of the assimilation depends on submatrices R_y^y extracted from the resolvent matrix of the linearised perturbation system (3). We will see in the next subsection an example of convergence of a purely forward assimilation.

2.2 Application to the linearised shallow-water equations

The non-linear shallow-water equations read

$$\frac{\partial \phi}{\partial t} + \nabla \, . \, (\phi \mathbb{V}) = o \qquad (11a)$$

$$\frac{\partial \mathbb{V}}{\partial t} + (\mathbb{V} . \nabla) \mathbb{V} + \nabla \phi + \mathbb{n} \times f \mathbb{V} = o \qquad (11b)$$

where ϕ and $\mathbb{V}$ are respectively the free-surface geopotential and horizontal velocity of a shallow fluid covering a horizontal domain S; $\mathbb{n}$ is a vertical unit vector, f the Coriolis coefficient, and ∇ the horizontal del operator. Linearised about the state of rest defined by

$$\phi = \Phi_o = c^{te} \tag{12a}$$

$$\mathbb{V} = o \tag{12b}$$

equations (11) become

$$\frac{\partial \delta\phi}{\partial t} + \Phi_o \, \delta \mathbb{V} = o \tag{13a}$$

$$\frac{\partial \delta \mathbb{V}}{\partial t} + \nabla \, \delta\phi + \mathbb{n} \times f \, \delta\mathbb{V} = o \tag{13b}$$

where $\delta\phi$ and $\delta\mathbb{V}$ are now "perturbations" from state (12). We are going to study the convergence of an assimilation when the appropriate linearised perturbation system is system (13). According to the developments of the previous subection, this is the case in either of the following situations

- the time evolution of the system under observation is given by the non-linear equations (11), the "observed solution" being the state of rest (12).

- the time evolution of the system under observation is given by the linear equations (13), the "observed solution" being any solution of these equations.

Despite their simplicity, considering these situations lead to a number of instructive conclusions which, as will be seen below, remain valid in more complex and more realistic cases.

It turns out not to be necessary to determine explicitly the matrix R_y^y corresponding to equations (13) in order to conclude as to the convergence of an assimilation. For any solution of equations (13), the following quadratic quantity

$$E = \frac{1}{2} \int_S (\phi^2 + \Phi_o \mathbb{V}^2) \, dS \tag{14}$$

which represents the total energy of the perturbation, is conserved in time (from now on, we will drop the prefixes δ in (13)). The geopotential Φ_o being necessarily positive in any physically meaningful situation, E is $\geqslant 0$ and can be 0 only if the perturbations ϕ and $\mathbb{V}$ are 0 everywhere on S. Let us consider an assimilation performed on observation of geopotential and/or velocity. In the simplest updating procedure the perturbation ϕ ($\mathbb{V}$) will be set equal to 0 over that part of S over which the geopotential (velocity) has been observed, and will be kept unchanged elsewhere. E cannot consequently increase at the time of an introduction of observations, and will tend to some limit $E_\infty \geqslant o$ as the assimilation is infinitely continued. If $E_\infty = o$, the assimilation will be successful in the sense that it will reconstitute the complete mass and wind fields of the observed solution. The following theorem is proved in T77.

- In the case of a forward-backward assimilation, or in the case of a purely forward assimilation performed on observations which have an exactly periodic time distribution, the limit E_∞ will be 0 for any initial perturbations ϕ and $\mathbb{V}$ if and only if the available observations are numerous enough to define uniquely the observed solution.

It is obvious that a necessary condition for convergence of an assimilation is that only one solution of the basic equations be compatible with the available observations. The above theorem states that in the case of equations (13), and because of energy conservation (14), this condition is also sufficient.

Energy conservation thus ensures the convergence of an assimilation performed with equations (13) but it tells us nothing as to the rapidity of that convergence. The latter will depend to a large extent on the particular space-time distribution of the observations. We will consider the case of successive observations of the complete mass field, separated by a constant time interval.

Assuming the Coriolis coefficient to be constant, Eqs. (13) read, for wave vector $\mathbb{k}$

$$\frac{d\phi}{dt} + \Phi_o D = 0 \qquad (15a)$$

$$\frac{dD}{dt} - \mathbb{k}^2\phi - f\zeta = o \qquad (15b)$$

$$\frac{d\zeta}{dt} + f D = o \qquad (15c)$$

where D and ζ are respectively the divergence and vorticity of the wind field. The resolvent matrix of system (15) between times t and $t + \Delta\tau$, which depends only on $\Delta\tau$, reads

$$R(\Delta\tau) = \begin{pmatrix} \gamma^2+(1-\gamma^2)\sin\beta & -\frac{\Phi_o}{\alpha}\sin\beta & -\frac{\Phi_o}{\alpha}\gamma(1-\cos\beta) \\ \frac{\mathbb{k}^2}{\alpha}\sin\beta & \cos\beta & \gamma\sin\beta \\ \frac{-\mathbb{k}^2}{\alpha}\gamma(1-\cos\beta) & -\gamma\sin\beta & 1-\gamma^2(1-\cos\beta) \end{pmatrix} \tag{16}$$

where α is the frequency of inertia-gravity waves

$$\alpha = \sqrt{f^2 + \mathbb{k}^2\,\Phi_o} \tag{17}$$

and β and γ are defined by

$$\beta = \alpha\Delta\tau \tag{18a}$$

$$\gamma = \frac{f}{\alpha} \tag{18b}$$

The parameter γ, which is always comprised between 0 and 1, is a measure of the ratio of the scale defined by $\mathbb{k}$ to the Rossby radius of deformation $\mathcal{R} = \frac{\sqrt{\Phi_o}}{f}$; γ is close to 1 for scales large compared to $\mathcal{R}$, and close to 0 for scales small compared to $\mathcal{R}$.

We will assume that the complete mass field has been observed at successive times separated by $\Delta\tau$, and that the observations are introduced into the model without modification of the wind field. The mass field ϕ and the wind field $\binom{D}{\zeta}$ therefore stand respectively for the vectors X and Y of Section 1. An important remark can be made at this point: since the geopotential appears in the divergence equation (15b), but not in the vorticity equation (15c), an introduction of mass observations performed without modifying the wind field will modify the first time derivative of D, but only the second derivative of ζ. We can therefore expect an assimilation of that type to generally have a stronger effect on the divergent component of the wind field than on its rotational component.

From formula (16) the expression for the matrix $R_Y^Y(\Delta\tau)$ is

$$R_Y^Y(\Delta\tau) = \begin{pmatrix} \cos\beta & \gamma\sin\beta \\ -\gamma\sin\beta & 1-\gamma^2(1-\cos\beta) \end{pmatrix} \tag{19}$$

We will successively consider the cases of a purely forward assimilation and of a forward-backward assimilation. Unless it is necessary to distinguish between different values of the argument, we will normally drop the argument $\Delta\tau$.

2.2.1 Forward assimilation

The wind difference is multiplied at each assimilation step by the matrix R_y^y. The rate of convergence will therefore depend on the spectral radius $\rho(R_y^y)$. The variations of $\rho(R_y^y)$ with the parameters β and γ are shown on Fig. 1. The spectral radius is everywhere $\leqslant 1$, as could a priori be deduced from the energy conservation (14). It is equal to 1 for the values $\beta = \ell\pi$ (ℓ integer), $\gamma = 0$, $\gamma = 1$ and for those values only which, in agreement with the theorem stated above, correspond to cases when the available observations do not uniquely define the observed solution. For $\beta = \ell\pi$, a "stroboscopic" effect between the intrinsic period $\frac{2\pi}{\alpha}$ of equations (15) and the observing period $\Delta\tau$ makes the successive observations redundant. For $\gamma = 1$, R_y^y is a rotation matrix of angle β (Eqn. (19)) whose both eigenvalues have modulus 1. Neither the divergence nor the vorticity is then reconstructed by the assimilation. For values of γ close to 1, both the divergence and the vorticity are reconstructed at a slow rate (see T77).

In a realistic barotropic model, the constant Φ_o must be about $9 \times 10^4\ m^2s^{-2}$. The value of γ is then approximately .3 for the largest scales, and tends to 0 as the scale decreases. It is therefore for small values of γ smaller than .3 that a detailed study of the assimilation is most relevant.

When $\gamma = 0$ (i.e. when $f = 0$), the rotational component of the wind is disconnected from the rest of the flow (Eqs. (15)). An assimilation will then reconstruct only the divergent component of the wind, as can be seen from expression (19). For small, non-zero values of γ, both the divergent and the rotational components will be reconstructed, but the former will be reconstructed more rapidly. After an introduction of mass observations, the proportion of the energy of the difference flow contained in the geostrophic mode is equal to

$$D = (1-\gamma^2)\ \frac{|\zeta|^2}{|\zeta|^2 + |D|^2}$$

It is at most equal to $1 - \gamma^2$. As the number of data introductions increases to infinity, and the difference between the model and the observed solutions tend to 0, D will either tend to a limit or oscillate between 0 and $1-\gamma^2$, depending on the values of β and γ. Fig. 2 shows that for small values of γ, D will tend to a limit which is close to 1. This means that the gravity wave component of the flow will be reconstructed much more rapidly than the geostrophic component. This is in complete contradiction with an often made hypothesis (see. e.g. Morel and Talagrand (1974)), according to which the main difficulty in data assimilation is to get rid of the spurious gravity waves excited by the introduction of observations. Indeed, what the assimilation will do in the present case is to reconstruct rapidly the gravity wave component of the flow

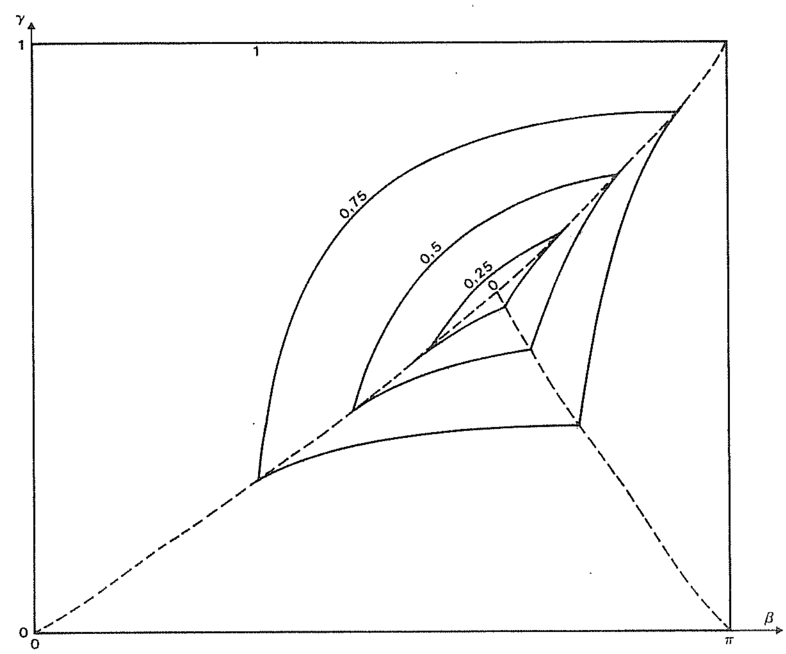


Fig.1 Variations of the spectral radius of the matrix R_y^y with respect to the parameters β and γ. Because of symmetries, only the range $0 \leqslant \beta \leqslant \pi$ is considered.

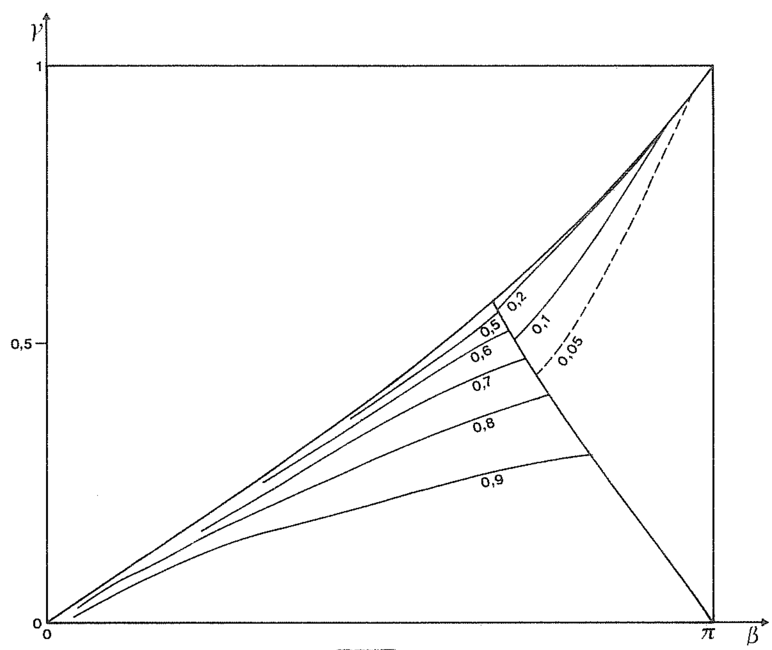


Fig. 2 Variations of the asymptotic value of the ratio D with respect to the parameters β and γ. In the blank area, which corresponds to $\gamma > \tan \frac{\beta}{4}$, D has no limit and oscillates between 0 and $1-\gamma^2$.

(possibly 0, if the observed solution is geostrophic) and slowly the geostrophic component.

Figs. 3a and 3b show the spectra of the divergence and vorticity differences respectively at the beginning and after a given time of forward assimilation. The assimilation was performed with a non-linear spherical barotropic model (described in T77), the "observed" solution being the state of rest (12), with $\Phi_o = 8.13 \times 10^4\ m^2 s^{-2}$. The features anticipated above are clearly apparent. Both the divergence and the vorticity are reconstructed at all wavelengths. The former is reconstructed more rapidly, especially at small wavelengths (small γ's). Wavelengths such that $\beta = \ell\pi$, ℓ integer, have been indicated. The decrease of the difference is slower for these wavelengths.

The effect of latitude on the assimilation is visible from Fig. 4, which shows the time variations of the divergence and vorticity differences, at both the equator and high latitudes. The divergence difference decreases rapidly, at a rate independent of latitude, while the vorticity difference decreases slowly at high latitude, and does not decrease at all at the equator, where the value of γ is 0.

2.2.2 Forward-backward assimilation

In one cycle of forward-backward assimilation performed on N successive observations of the mass field separated by $\Delta\tau$, the wind difference $\binom{D}{\zeta}$ is multiplied N-1 times by $R_Y^Y(-\Delta\tau)$ in the backward phase of the cycle, and N-1 times by $R_Y^Y(\Delta\tau)$ in the forward phase. The amplification matrix A of (10) is therefore

$$A = [R_Y^Y(\Delta\tau)]^{N-1}\,[R_Y^Y(-\Delta\tau)]^{N-1}$$

From the energy conservation (14), we can tell that necessarily $\rho(A) \leqslant 1$. For $N > 2$ (if $N=2$, $\rho(A) = 1$ for all values of β and γ because the uniqueness condition stated above is not satisfied), the variations of $\rho(A)$ with β and γ are similar to those shown on Fig. 1. In particular $\rho(A) = 1$ for the same limiting values. The conclusions already drawn remain valid.

An interesting question is the following: for a given time distribution of observations, will the convergence be more rapid in a forward-backward assimilation, or in a purely forward assimilation? In one cycle of forward-backward assimilation, the wind difference is multiplied by the above matrix A. In the same time, i.e. $2(N-1)\Delta\tau$, of forward assimilation, the wind difference is multiplied by $C = [R_Y^Y(\Delta\tau)]^{2(N-1)}$. The question considered is therefore equivalent to comparing the spectral radii $\rho(A)$ and $\rho(C)$. Eqs. (18a) and (19) show

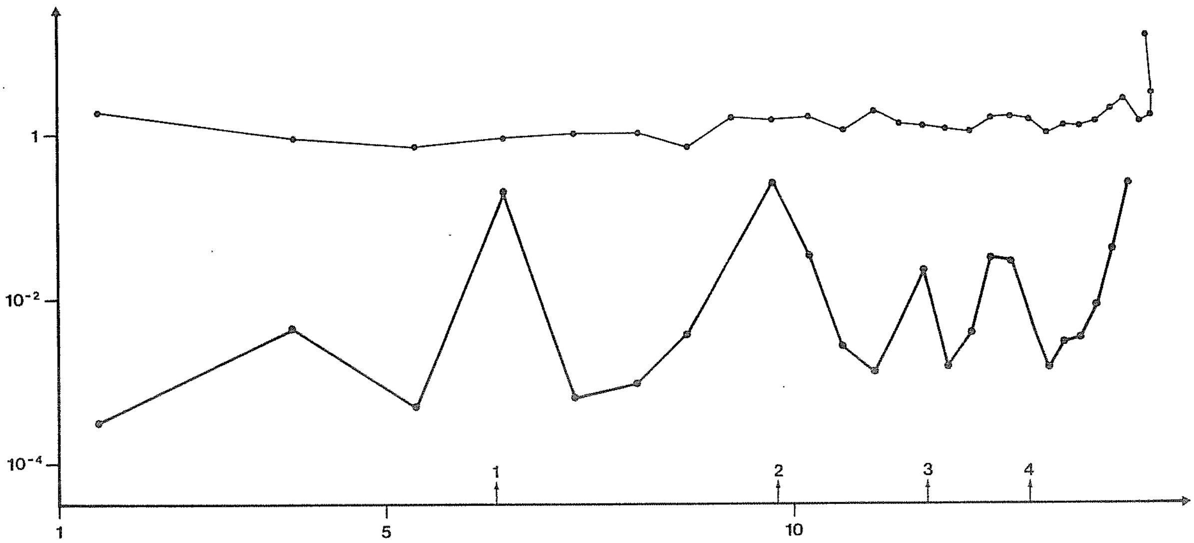


Fig. 3a Spectra of the divergence difference at the beginning (upper curve) and after 9 days of assimilation (lower curve). The unit in the vertical axis is arbitrary, but the initial difference has been generated by a random perturbation with standard deviation .58 ms^{-1} on each component. The abscissa variable is the logarithm of the modulus of the Laplacian eigenvalue. It mostly depends on the latitudinal wavenumber of the corresponding eigenfunction. The arrows indicate abscissa values such that $\beta=\ell\pi$, ℓ integer, together with the corresponding values of ℓ.

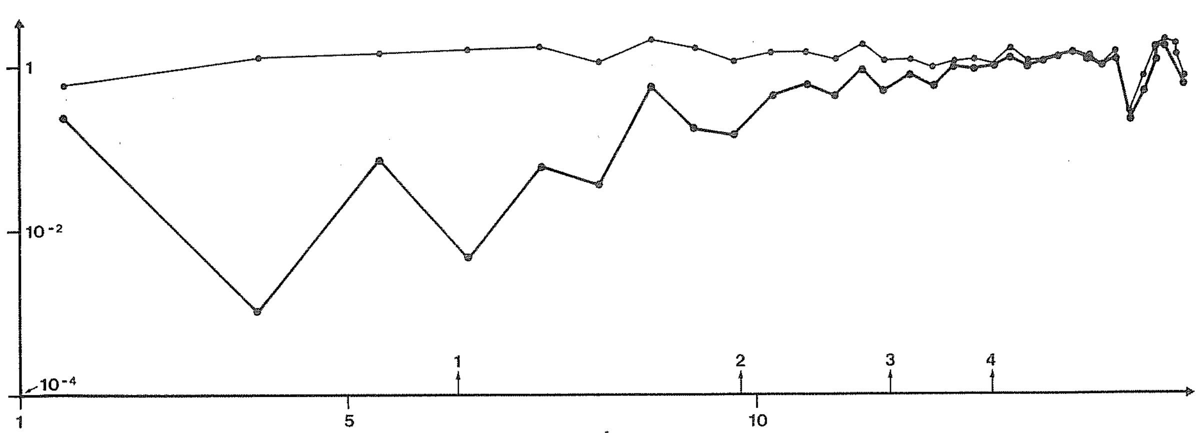


Fig.3b Same as 3a, for the vorticity difference

that $R_y^y(-\Delta\tau)$ is the transpose of $R_y^y(\Delta\tau)$, so that A and C are respectively equal to $B\tilde{B}$ and B^2, where B is the matrix $[R_y^y(\Delta\tau)]^{N-1}$ and $\tilde{B}$ is the transpose of B. Now a result of matrix algebra (see. e.g. Varga (1962)) tells us that, for any matrix B, $\rho(B\tilde{B})$ is the square of the Euclidean norm of B. Consequently

$$\rho(B^2) \leqslant \rho(B\tilde{B}) \tag{20}$$

which, in the present case, means that a purely forward assimilation will converge at least as rapidly as a forward-backward assimilation.

Fig. 5 shows the time variations of the rms wind differences in assimilations of both types performed on data with the same time distribution. In the beginning, the decrease is more rapid for the purely forward assimilation, in agreement with inequality (20). Later on, however, the difference starts increasing in the forward assimilation. This increase, which is in contradiction with the energy conservation (14) can be traced to the fact that energy is conserved by the model only to first order with respect to the time discretisation increment.

2.3 Generalisation to the linearised primitive equation

The foregoing results can be extended to the linearised multi-level primitive equations. These equations conserve with time a positive definite quadratic energy, which is a function of surface pressure, potential temperature and wind (see T77). This energy plays exactly the same role as (14) and it results that an assimilation performed on observations of surface pressure, potential temperature and/or wind will converge under the only condition that the available observations uniquely define the corresponding solution.

In the case of observations of the complete mass field (or wind field), the linearised primitive equations can be separated into their vertical modes, the time evolution of each of which is given by Eqs. (15), Φ_o being replaced by the appropriate equivalent depth. The equivalent depth decreases, and the parameter γ consequently increases (Eqs. (17) and (18b)) as the order of the vertical mode increases. According to the analysis of the previous subsection, we can therefore expect that the difference between the rates of reconstitution of divergence and vorticity will be less marked for modes of higher order. Figs. 6a and 6b, which are in the same format as Figs. 3a and 3b refer to the internal mode of a two-level model, for which the equivalent depth was $\Phi_1 = 1400\ m^2s^{-2}$. At large scales, divergence and vorticity are reconstituted at about the same rate. At small scales divergence is still reconstituted more rapidly than vorticity, but the latter is reconstituted more rapidly than on the external mode (see Fig. 3a).

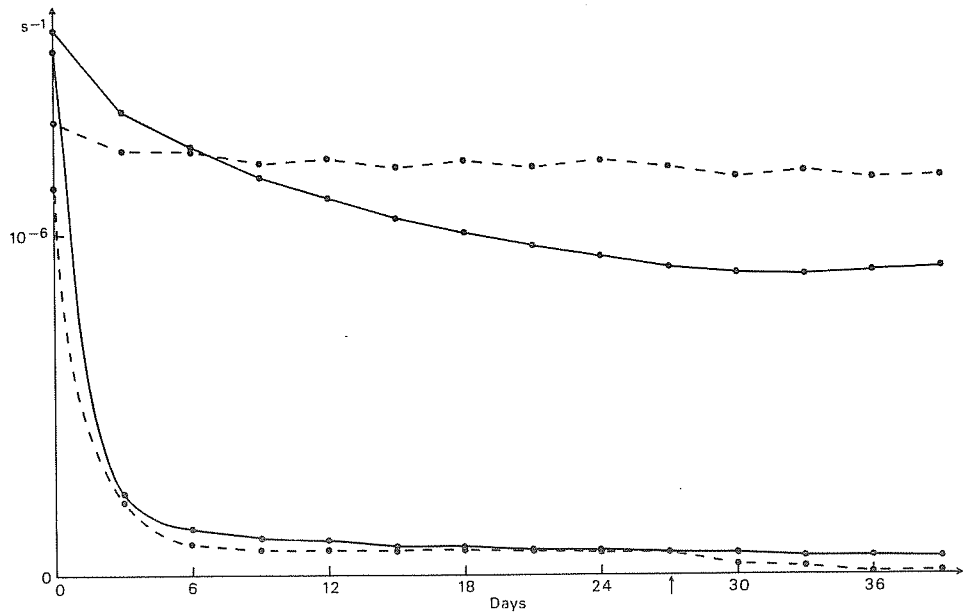


Fig.4 Variations with time of the rms differences of
- high latitude divergence (lower solid curve)
- equatorial divergence (lower dotted curve)
- high latitude vorticity (upper solid curve)
- equatorial vorticity (upper dotted curve)

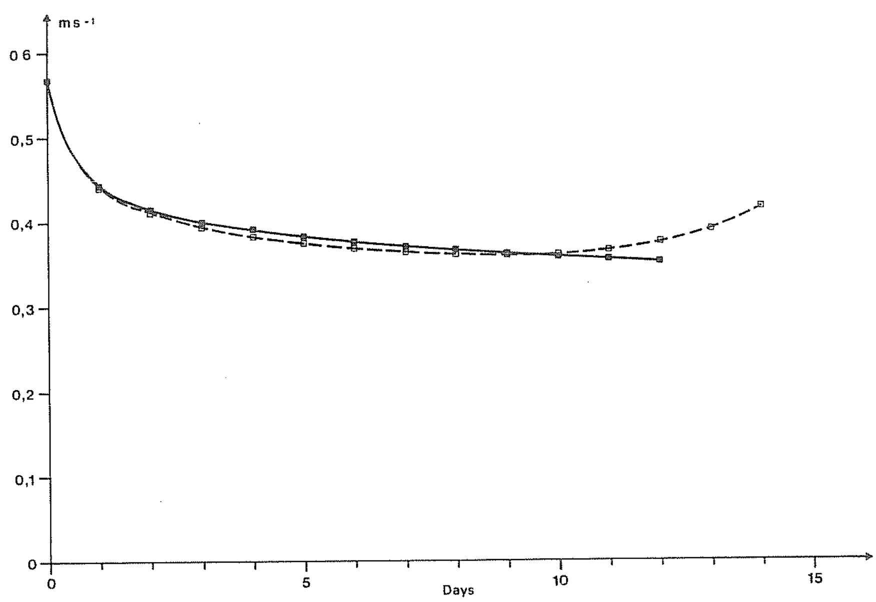
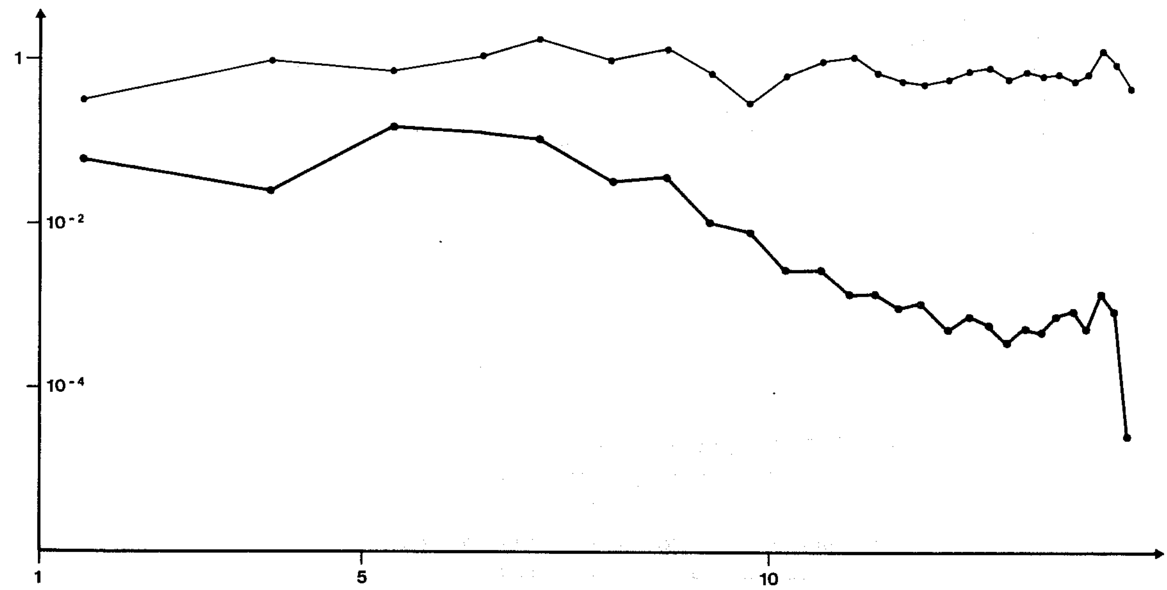
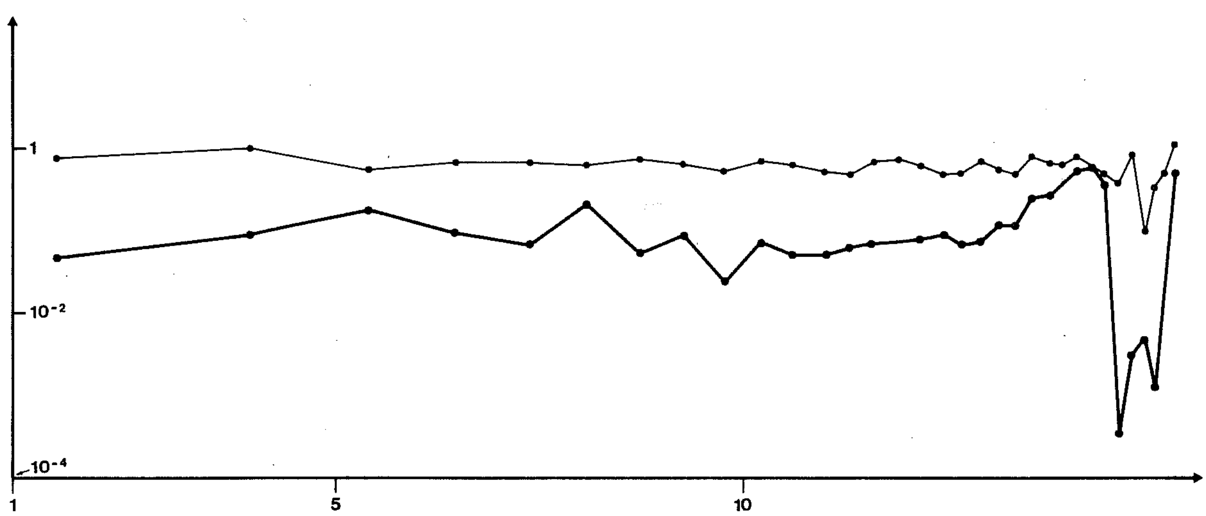


Fig.5 Variations of the rms wind difference in a forward-backward assimilation (solid curve) and a purely forward assimilation (dotted curve), both with the same space-time distribution of observations.



6a Divergence



6b Vorticity

Fig. 6 Same as Fig. 3 for the internal mode of a two-level model

2.4 The effect of geostrophic adjustment

The results established so far are valid for any solution of equations (15), whether this solution is geostrophic or not. It has often been hypothesised that geostrophic adjustment, whose effect is to reestablish geostrophic balance after it has been disrupted by an introduction of observations, is a necessary ingredient for the success of an assimilation procedure. Our results show that such is not the case. We will come back later to this point, and will here study how the results established so far are modified by the presence of a geostrophic adjustment process. We will assume this process (whose exact mechanism is irrelevant) to be such that after each introduction of observations, and before the next introduction, the gravity wave component of the flow is completely damped, while the geostrophic component is not altered in any way. This simple "model" of geostrophic adjustment, has already been used by several authors (e.g. by Williamson and Dickinson (1972)).

The energy E of the difference flow will now decrease both when observations are introduced, as before, and between introduction because of the damping of gravity waves. However, since the energy decrease can be distributed between the various components of the flow, this decrease will not necessarily be more rapid than in the absence of geostrophic adjustment.

Acting on the difference ($\phi = 0, D, \zeta$) resulting from an introduction of observations, geostrophic adjustment will leave only the corresponding geostrophic components, viz.

$$\phi_g = -\frac{\Phi_o}{\alpha}\gamma\zeta$$

$$D_g = 0$$

$$\zeta_g = (1-\gamma^2)\zeta$$

This component is stationary with time, and the next introduction of observations will result in ϕ_g being set equal to 0. The wind difference $\begin{pmatrix} D \\ \zeta \end{pmatrix}$ will therefore be multiplied at each assimilation step by the following matrix

$$G = \begin{pmatrix} 0 & 0 \\ 0 & 1-\gamma^2 \end{pmatrix} \qquad (21)$$

The spectral radius of G, equal to $1-\gamma^2$, is always less than 1, except for γ=o. Fig. 7 shows the variation of the difference $\rho(R_y^y) - \rho(G)$ with β and γ.

- for large values of γ, this difference is positive, which means that an assimilation will converge more rapidly if geostrophic adjustment is present. This can be easily explained: for these values of γ (i.e. for scales large compared with the Rossby radius of deformation) the effect of geostrophic adjustment is to adjust the wind field to the mass field, the latter not being modified. Geostrophic adjustment can therefore do what simple assimilation cannot do in this particular case, namely reconstruct a wind field consistent with the observed mass field.

- for small values of γ on the contrary (which are the relevant values for an atmospheric barotropic model), and except in the vicinity of the values β = o and $\beta = \pi$, an assimilation is more efficient if no geostrophic adjustment is present. This too can easily be explained: for small values of γ, geostrophic adjustment results in the mass field being adjusted to the wind field, so that mass observations are in effect rejected. For these values, an assimilation without geostrophic adjustment converges slowly, but the presence of a geostrophic adjustment only makes the convergence still slower.

3. COMPARISON WITH RESULTS OBTAINED WITH THE FULLY NON-LINEAR EQUATIONS

It is interesting to compare the theoretical results obtained above with numerical results obtained using a fully non-linear model. Figs. 8 and 9, which are in the same format as Figs. 3 and 6, are relative to an assimilation performed with non-linear equations. The model used, and the space time distribution of observations, were the same as before, but the "observed" solution was a fully non-linear meteorologically realistic solution. In addition, the model contained a divergence dissipation, acting as geostrophic adjustment (see Sadourny (1972)).

Figs. 8 and 9 refer respectively to the external and internal modes of the model. The main qualitative features deduced from Eqn. (15) are still present. Both the divergence and the vorticity are reconstructed at all wavelengths and for both vertical modes. The divergence is reconstructed more rapidly, especially at small wavelengths and for the external mode. All stroboscopic effects have disappeared, which corresponds to the fact that non-linearity destroys any periodicity in the flow. There is moreover an important change from the linear case: the global rate of reconstitution of the wind field is now much more rapid (Fig. 10). This fact may be due in part to the dissipation

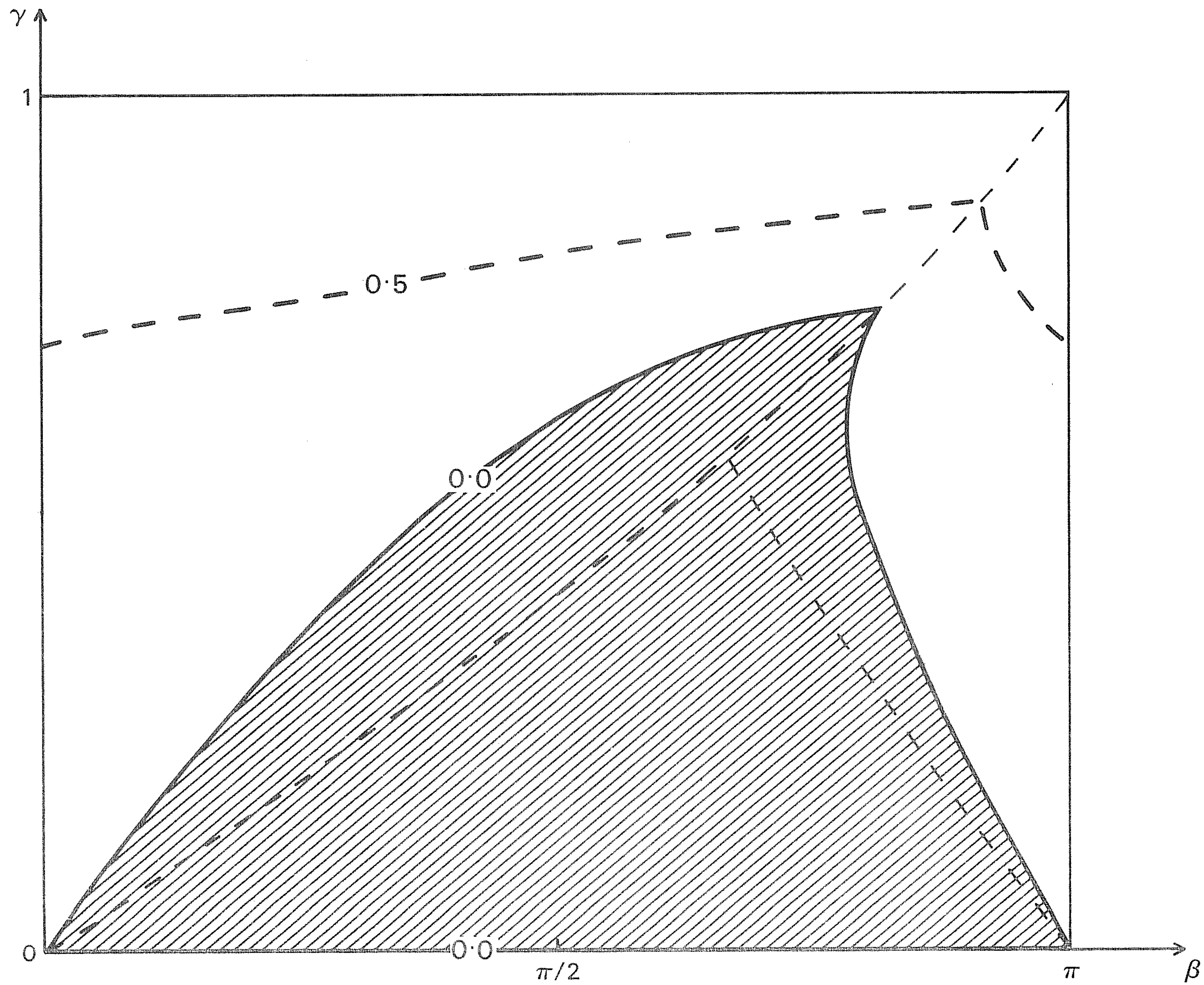
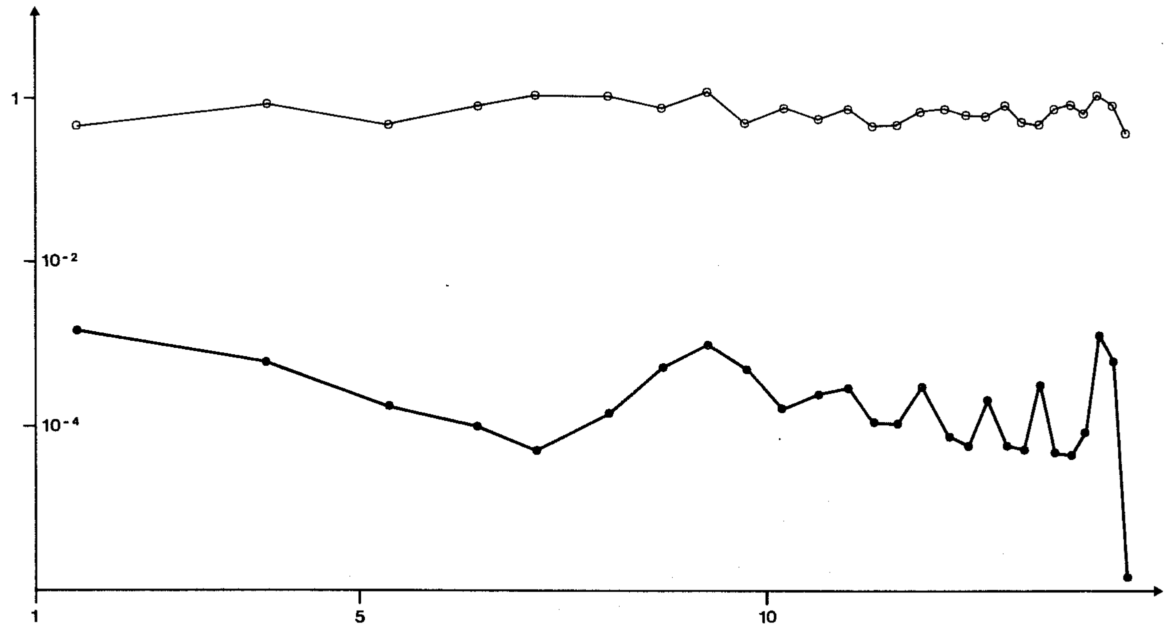
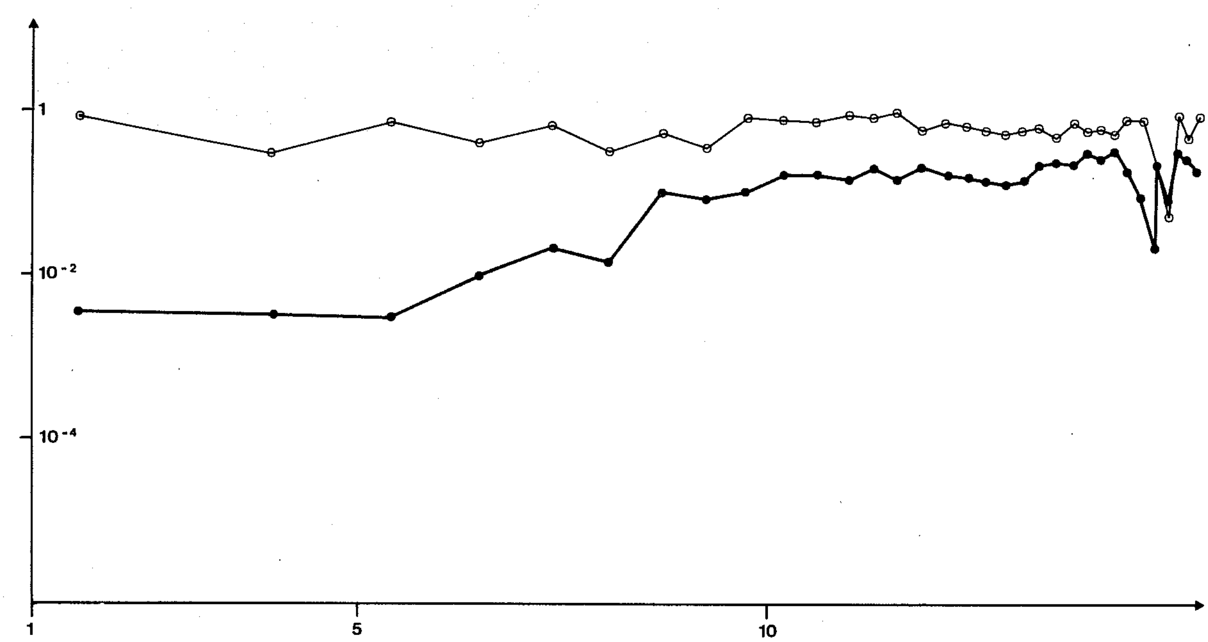


Fig. 7 Variations of the difference $\rho(R_y^y)-\rho(G)$ with respect to the parameters β and γ .
The difference is negative in the shaded area, positive elsewhere.

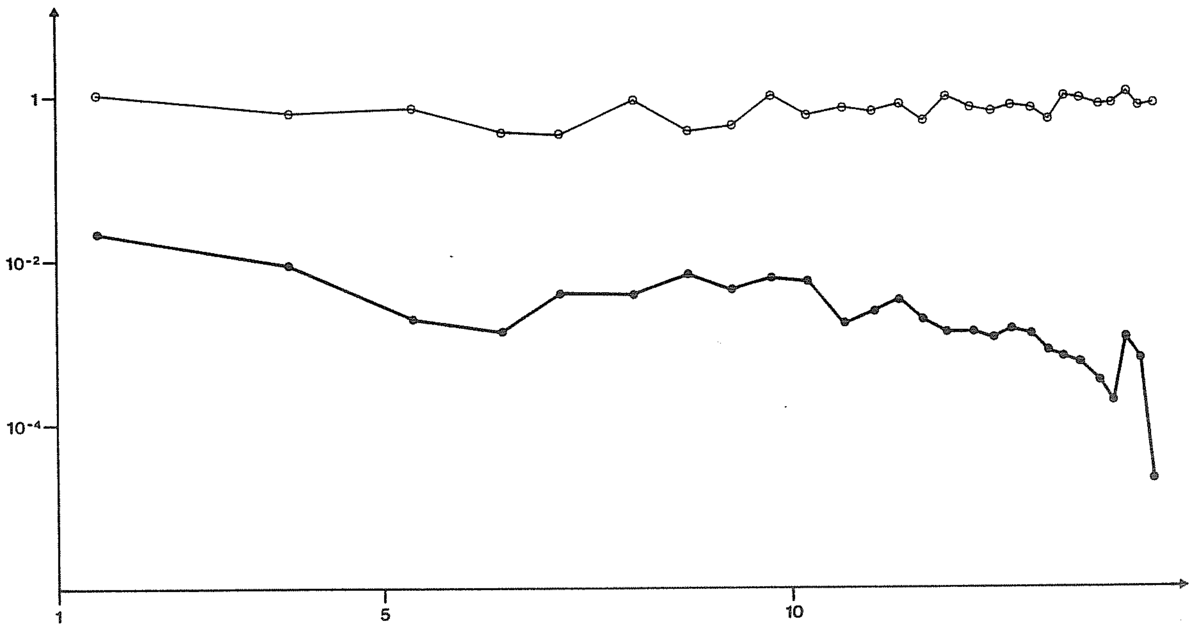


8a Divergence

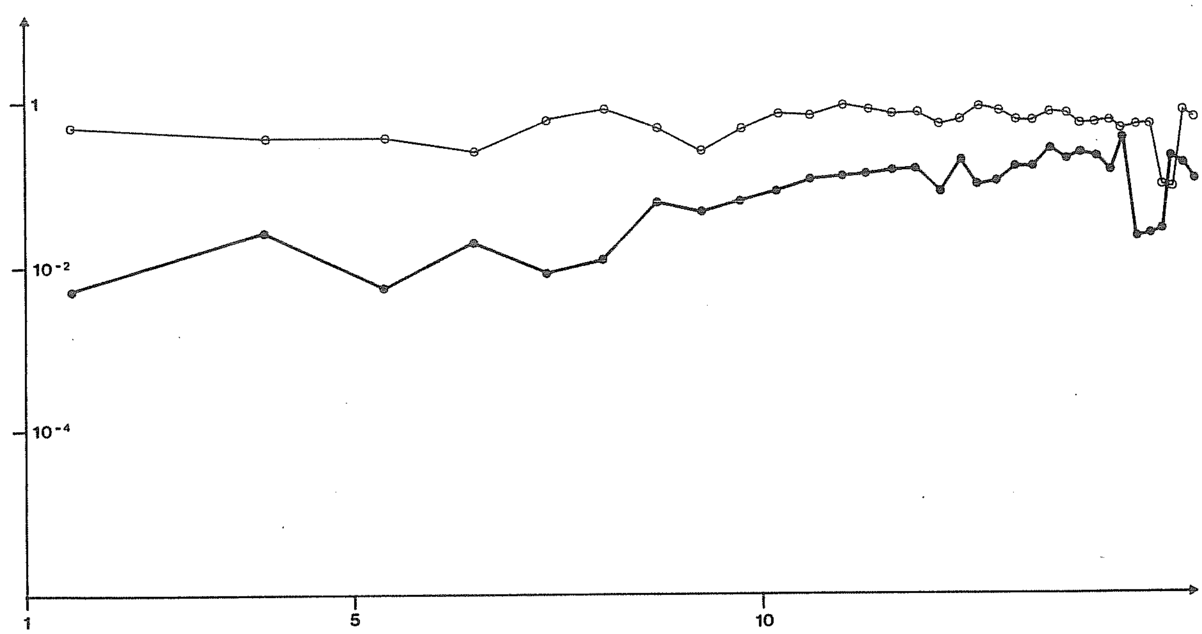


8b Vorticity

Fig. 8 Same as Fig. 3 for the external mode of a fully non-linear solution



9a Divergence



9b Vorticity

Fig. 9 Same as Fig. 3 for the internal mode of a fully non-linear solution of gravity waves, but cannot be due only to that dissipation, since it remains true for small values of the parameter γ, for which, as seen above, dissipation of gravity waves will not speed up the rate of convergence. For instance, the equatorial vorticity is now reconstructed (Fig.11) while it was not in the linear case (Fig. 4). Non-linear advection must therefore be at least in part responsible for the speeding up of convergence.

4. THE CASE OF SUCCESSIVE OBSERVATIONS CLOSE IN TIME

The explicit determination of the amplification matrix (10) and/or of its spectral radius will most often be difficult, if not impossible, since it will normally require the explicit knowledge of the linearised perturbation system (3) in the vicinity of the observed solution. In the case of the meteorological equations, it is unlikely that system (3) can be determined, except for a few explicitly known solutions. However, when the successive observations are close in time, the amplification matrix and the corresponding convergence criterion assume simple forms, depending only on conditions local in time. We will now proceed to the study of this simpler case.

4.1 The simplified convergence criterion

We will first assume that two observations of X are available, at times t_1 and t_2. The amplification matrix is

$$A = R_y^y\ (t_2,t_1)\ R_y^y\ (t_1,t_2)$$

where we do not use the auxiliary time τ any more. The reversibility equation (8) together with decomposition (5) imply

$$R_y^y(t_2,t_1)\ R_y^y(t_1,t_2)\ +\ R_y^x(t_2,t_1)\,R_x^y(t_1,t_2)\ =\ I$$

where I is now the unit matrix of order q.

Therefore

$$A = I - R_y^x(t_2,t_1)\ R_x^y(t_1,t_2) \tag{22}$$

Equations (3) imply in turn

$$R_y^x(t+\Delta t,t)\ =\ \Delta t\ \frac{DG}{DX}\ (t)\ +\ o(\Delta t)$$

$$R_x^y(t+\Delta t,t)\ =\ \Delta t\ \frac{DF}{DY}\ (t)\ +\ o(\Delta t)$$

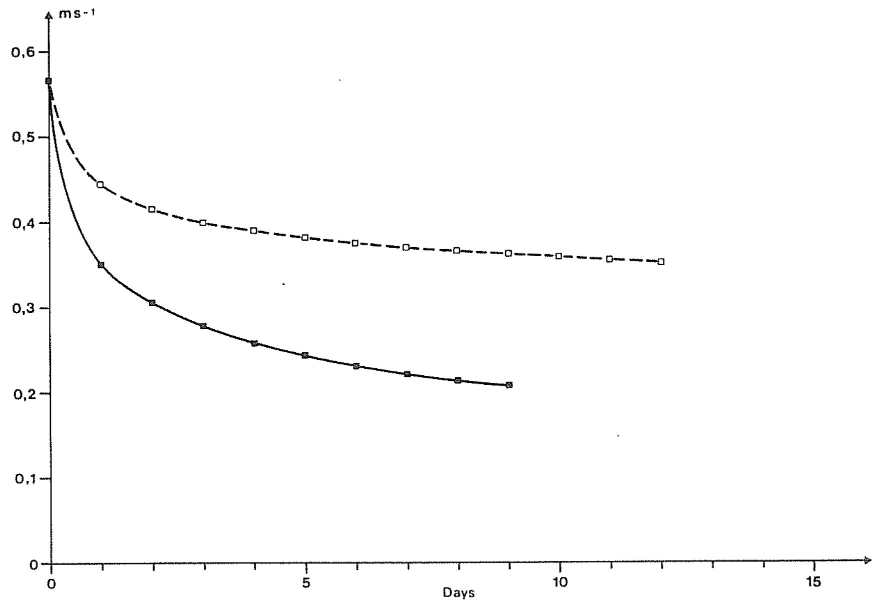


Fig. 10 Variations with time of the global wind rms difference in the non-linear (solid curve) and linear (dotted curve) cases respectively.

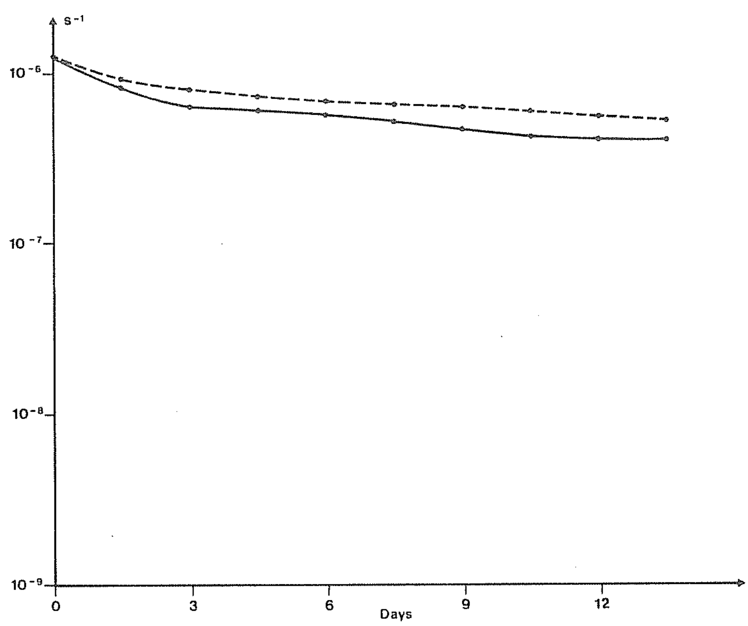


Fig. 11 Variations of the rms equatorial vorticity difference in the non-linear case. Solid curve: external mode. Dotted curve: internal mode.

Setting $\Delta t = t_2 - t_1$, and carrying these expressions into (22) lead to

$$A = I + \Delta t^2 \frac{DG}{DX}(t_1) \frac{DF}{DY}(t_2) + o(\Delta t^2)$$

To order Δt, the matrices $\frac{DG}{DX}$ and $\frac{DF}{DY}$ can be taken indifferently at time t_1 or t_2.

We shall write

$$A = I + \Delta t^2 \frac{DG}{DX} \frac{DF}{DY} + o(\Delta t^2) \tag{23}$$

with no more precision.

Let λ_j (j=1,...,q) be the eigenvalues of the qxq matrix $\frac{DG}{DX} \frac{DF}{DY}$. The eigenvalues of A are

$$a_j = 1 + \lambda_j \Delta t^2 + o(\Delta t^2) \qquad (j=1,\dots,q)$$

The convergence criterion derived above is that in the complex plane all the eigenvalues a_j lie inside the circle (C) centered at the origin with radius unity. This condition is verified if all the eigenvalues λ_j have a strictly negative real part (Fig. 12).

$$\mathcal{R}(\lambda_j) < o \qquad (j=1,\dots,q) \tag{24}$$

The matrix $\frac{DG}{DX} \frac{DF}{DY}$ is the product of two matrixes, each of which represents the dependence of the time evolution of one of the two parts X or Y with respect to the other. The matrix $\frac{DG}{DX} \frac{DF}{DY}$ therefore represents in some sense the coupling between the time evolutions of X and Y. It will be called the coupling matrix between X and Y.

In the case when X has been observed at N + 1 successive times separated by a constant time interval Δt, it can be shown (e.g. by induction on N) that the amplification matrix is

$$A = I + N\, \Delta t^2 \frac{DG}{DX} \frac{DF}{DY} + o(\Delta t^2) \tag{25}$$

which leads to the same convergence criterion (24).

4.2 Application to the shallow-water equations

Let us consider the shallow-water equations (11), the geopotential ϕ and the velocity $\mathbb{V}$ standing respectively for the vectors X and Y, and equations (11a) and (11b) for equations (1a) and (1b) respectively. For the previous developments to be valid, we must assume equations (11) to have been discretised to a finite number of parameters. However, for the sake of simplicity, we will keep notations usual for a continuum.

The velocity field $\mathbb{V}$ appearing linearly in Eq. (11a), the equivalent of the jacobian matrix $\frac{DF}{DY}$ is now the linear operator L_1 which to any "perturbation" $\partial\mathbb{V}$ of the wind field, associates the scalar field defined by

$$L_1(\partial\mathbb{V}) = -\nabla.(\phi\ \partial\mathbb{V})$$

Similarly, the geopotential ϕ appears linearly in Eq. (11b) and the equivalent of the jacobian matrix $\frac{DG}{DX}$ is the linear operator L_2 which, to any perturbation $\partial\phi$ of the geopotential field, associates the vector field defined by

$$L_2(\partial\phi) = -\nabla(\partial\phi)$$

The equivalent of the coupling matrix $\frac{DG}{DF}\frac{DF}{DY}$ is the linear operator obtained by composing L_1 with L_2, i.e. the operator C which, to any wind field perturbation $\partial\mathbb{V}$ associates the following vector field

$$C(\partial\mathbb{V}) = L_2[L_1(\partial\mathbb{V})] = \nabla[\nabla.(\phi\partial\mathbb{V})] \tag{26}$$

The next step is to determine the signs of the real parts of the eigenvalues of C. First we introduce the scalar product defined for any two wind field perturbations $\partial\mathbb{V}$ and $\partial\mathbb{V}'$ (with possibly complex components) by

$$<\partial\mathbb{V}, \partial\mathbb{V}'> = \int_S \phi\ \partial\mathbb{V}.\partial\mathbb{V}'^*\ dS \tag{27}$$

where $\partial\mathbb{V}'^*$ is the complex conjugate of $\partial\mathbb{V}'$. Since the geopotential is positive everywhere on the domain S, (27) defines a scalar product. Then, for any two two perturbations $\partial\mathbb{V}$ and $\partial\mathbb{V}'$

$$<\partial\mathbb{V}, C(\partial\mathbb{V}')> = \int_S \phi\ \partial\mathbb{V}.\nabla[\nabla.(\phi\partial\mathbb{V}'^*)]\ dS = -\int_S \nabla.(\phi\partial\mathbb{V})\ \nabla.(\phi\partial\mathbb{V}'^*)\ dS \tag{28}$$

The latter expression is symmetrical with respect to $\delta\mathbb{V}$ and $\delta\mathbb{V}'^*$, which means that the operator C is self-adjoint with respect to the scalar product (27).

Its eigenvalues are consequently real. Let λ be one of these eigenvalues. Setting $\delta \mathbb{V}' = \delta \mathbb{V}$ in (28), where $\delta \mathbb{V}$ is an eigenfunction associated with λ, leads to

$$\lambda < \partial \mathbb{V}, \partial \mathbb{V} > = - \int_S [\nabla . (\phi \ \partial \mathbb{V})]^2 \ ds$$

which shows that λ is negative except if

$$\nabla . (\phi \ \partial \mathbb{V}) = 0 \quad \text{everywhere on S} \tag{29}$$

It is obvious from (28) that any perturbation $\partial \mathbb{V}$ which satisfies this condition is an eigenfunction of C, associated with eigenvalue 0. All the eigenvalues of C are therefore negative, except one which is 0.

If we ignore for the time being the $o(\Delta t^2)$ term in (23), it results than an assimilation performed on successive observations of the geopotential separated by Δt will reconstitute the windfield except for a residual error verifying (29). More precisely, taking the vorticity of (26) shows that the vorticity of the wind difference is not modified in an assimilation cycle. The residual error $\delta \mathbb{V}_\infty$ will therefore be defined by the following two conditions: it satisfies (29) and its vorticity is equal to the vorticity of the wind difference at the beginning of the assimilation.

These results are true only if Δt is small enough so that no eigenvalue of the amplification matrix (23) has modulus larger than 1. Still ignoring the $o(\Delta t^2)$ term, this condition reads

$$- 1 < 1 + \lambda_M \ \Delta t^2$$

or

$$\Delta t < \sqrt{- \frac{2}{\lambda_M}} \tag{30}$$

where λ_M is the (negative) eigenvalue of C with the largest modulus. It is not difficult to see that condition (30) is basically the same as the Courant-Friedrichs-Lewy condition for stability of a numerical integration of Eqs. (11). It leads for Δt to values of the same magnitude, typically $\Delta t < 15$ mn for ordinary spatial resolutions.

The additional term $o(\Delta t^2)$ in (23) turns out to be too complex to be studied analytically, and numerical experiments have been performed in order to determine if, and how, it modifies the above results. Since the total number of parameters of a shallow-water equation model is three times the number of parameters defining the complete mass field, two successive observations of the latter cannot in any case define the complete state of the flow. Accordingly, the experiments were performed with three successive observations of ϕ, separated by Δt. Development (23) is then replaced by development (25) with N=2. Condition (30) is replaced by the still stricter condition

$$\Delta t < \sqrt{\frac{1}{\lambda_M}} \equiv \Delta t_c \qquad (31)$$

For the particular model used in the present experiments, the value of Δt_c was about 12.6 mn. The time discretisation increment used was

$$\Delta\tau = 12 \text{ mn} \simeq .95\ \Delta t_c$$

As for the time interval Δt between successive observations of ϕ, two values were used, one

$$\Delta t = \Delta\tau$$

which satisfies condition (31), and the other

$$\Delta t = 3\ \Delta\tau \simeq 2.85\ \Delta t_c$$

which does not.

Fig. 13 shows the variations of the root-mean-square wind difference in the two experiments. The difference decreases in both cases and more rapidly, for the same number of assimilation cycles, with the larger value of Δt. This means that, when Δt reaches the limit value Δt_c, the $o(\Delta t^2)$ term is no more negligible, and indeed is such as to decrease further the spectral radius of the amplification matrix. An additional fact of interest can be seen from Fig. 14, which shows the variations of the rms vorticity difference in the same two assimilations. As said above, the vorticity is not modified by the first two terms of development (25), and only the $o(\Delta t^2)$ term can account for a possible variation of the vorticity difference. Fig. 14 shows that this difference decreases for both values of Δt, and more rapidly for the larger value.

It thus appears that, at least in the case of the numerical model used here, the $o(\Delta t^2)$ term contributes to the convergence of an assimilation and, particularly, leads to the reconstitution of the rotational part of the wind field. The basic mathematical reason for this has not been rigorously

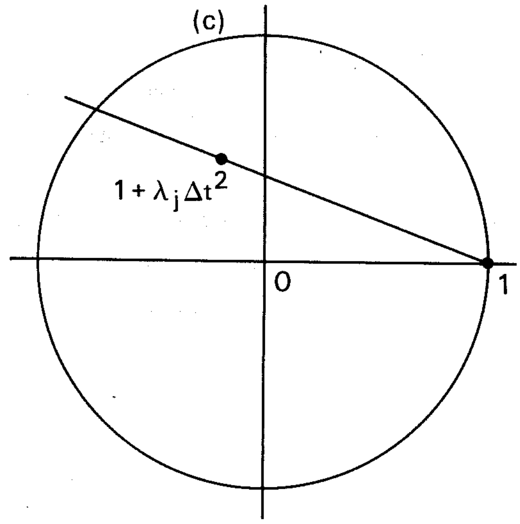


Fig. 12 Graphical illustration of condition (24). For small Δt, the point $1 + \lambda_j \Delta t^2 + o(\Delta t^2)$ lies inside circle (C) if the real part $\Re(\lambda_j)$ of λ_j is strictly negative, and outside (C) is $\Re(\lambda_j)$ is strictly positive. When $\Re(\lambda_j) = 0$, the point may lie inside, on or outside circle (C), depending on $o(\Delta t^2)$

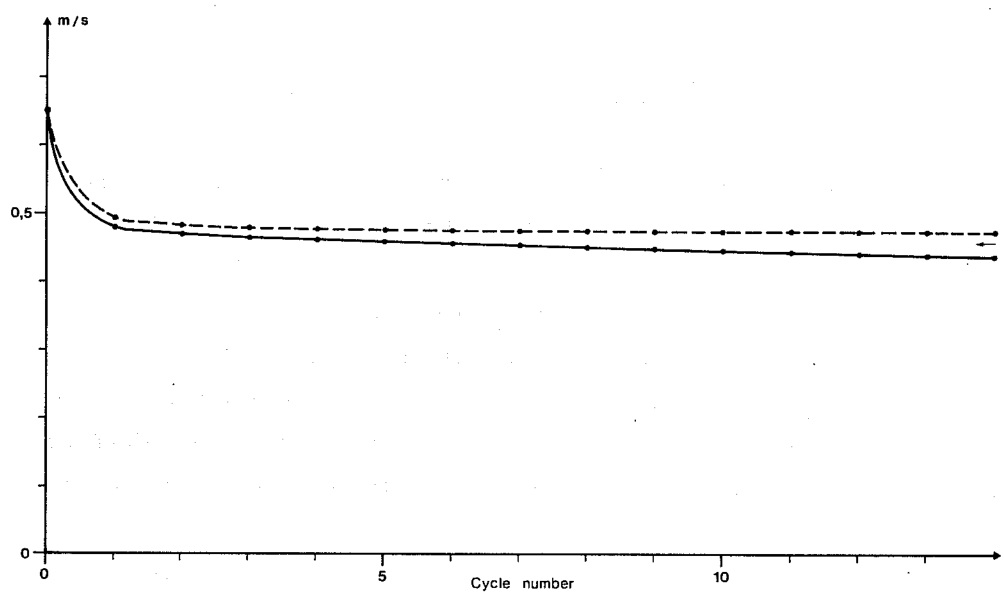


Fig. 13 Variations, as functions of the number of assimilation cycles, of the rms wind difference in two assimilations of mass observations. The upper curve corresponds to a time interval between observations $\Delta t = \Delta t_c$, and the lower curve to $\Delta t = 3\Delta t_c$.

established, but two facts strongly suggest that the role of the Coriolis acceleration is here fundamental. First, it has been shown in Section 2 that, in the case of the linearised equations, it is the Coriolis acceleration, which is then the only interaction between divergence and vorticity, which ensures the reconstitution of vorticity. The second fact is apparent from Fig. 15, which shows, as functions of latitude, the proportion of the initial rms difference remaining after a given time of assimilation on divergence and vorticity respectively (this figure refers to the assimilation performed with the larger value $\Delta t = 3\Delta\tau$). The rate of reduction of the divergence difference is rapid and exhibits no variation with latitude. The rate of reduction of the vorticity difference, on the contrary, is slow, particularly in low latitudes. This suggests that the reconstitution of vorticity depends to a large extent on the Coriolis parameter. It must be noted however that the vorticity difference is reduced at all latitudes, including at the equator. In Section 2, the same fact was observed only with non-linear equations, and was ascribed to an additional effect of advection. The same explanation probaly applies also here.

It is noteworthy that, as in the case of the linear equations, the numerical convergence of an assimilation procedure is independent of any specific feature of the solution to be reconstituted and, in particular, of whether or not this solution is geostrophic. It does not require either the presence in the assimilating model of any dissipative process, intended at restoring the geostrophic balance after it has been disrupted by the introduction of observations. These facts are obvious for the theoretical results, which have been derived without any particular hypothesis about the "observed" solution, and without assuming the presence of any dissipative process. As for the numerical results just presented, they have been obtained from an "observed" solution which had been produced by numerical integration of the inviscid equations (11). For some reason, this integration produced a rather large amount of gravity waves. This in no way prevented the reconstitution of both the divergent and rotational parts of the wind field.

Because of the particular conditions under which the results presented in this Section have been obtained (small time interval between successive observations, no dissipation) they are not directly relevant to the practical problem of assimilation. But, together with the results obtained for the linear equations, they provide a clear description of the processes at play in an assimilation of mass observations. First, the direct influence of data introduction upon the divergence results in a rapid reconstitution of the latter. The

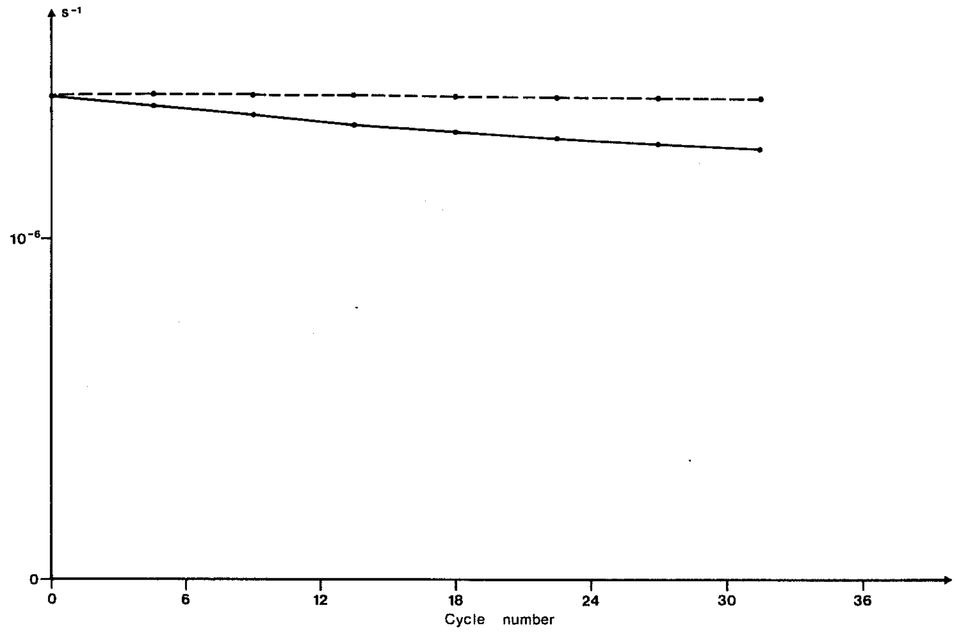


Fig. 14 Same as Fig. 13, for the variations of the rms vorticity difference

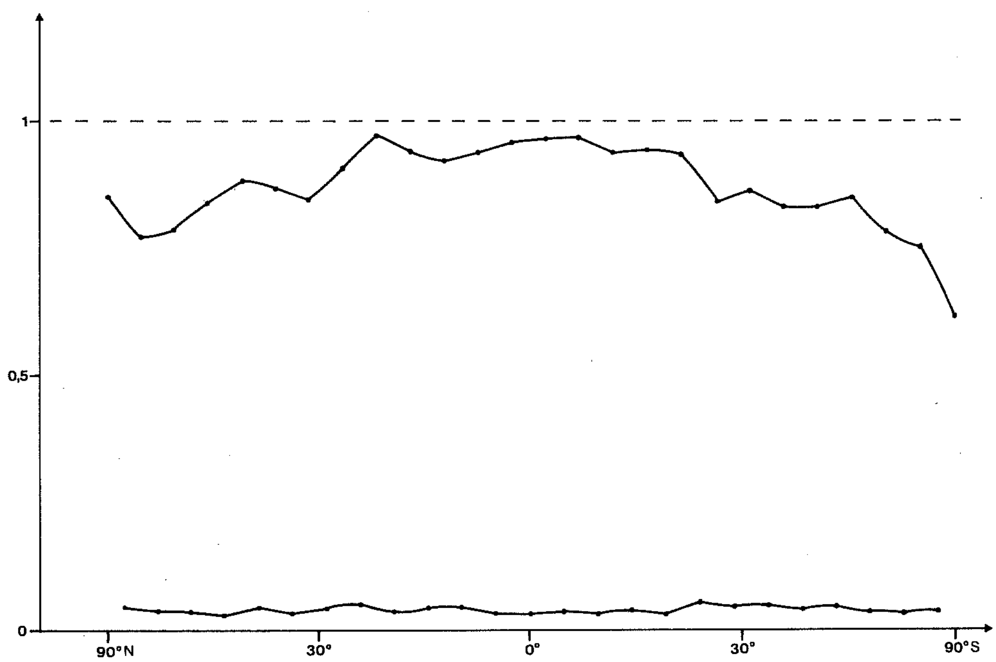


Fig. 15 Latitudinal distribution of the proportion of initial rms difference remaining after a given time of assimilation.
Upper curve: vorticity rms difference.
Lower curve: divergence rms difference

vorticity field is also reconstructed, but indirectly and more slowly, through Coriolis acceleration and, to a lesser extent, through advection.

As for the somewhat unexpected fact that neither geostrophicity nor the presence of a geostrophic adjustment process are basically necessary for the convergence of an assimilation, two conclusions at least can be drawn from it.

- a usual interpretation is that it is because of geostrophic adjustment that assimilation processes converge at all. Our results show that this interpretation is not correct. Other effects than geostrophic adjustment play a role in assimilation, and these effects must be taken into account in the definition of assimilation procedures.

- an assimilation can relatively easily reconstruct the divergence field from the observed time evolution of the mass field. This is clearly apparent, for instance, from Fig. 15. Indeed, the time evolution of the mass field contains information on the divergence inaccessible through direct measurement. For instance, a surface pressure tendency of 1 mb hr^{-1} corresponds to a vertically averaged divergence of $3 \times 10^{-7}\ s^{-1}$, well beyond the accuracy of direct measurements. In all present assimilation procedures, the final values of the divergence are determined mostly by the initialization step, which is intended at suppressing unrealistic gravity waves. But it is not known to which accuracy the real values of the divergence are reconstructed by the initialization. It would be of interest to define a method for reconstructing the divergence with all the accuracy allowed by the observations of the mass field. The results presented above suggest that this is possible.

5. THE EFFECT OF OBSERVING AND MODELLING ERRORS

We have exclusively considered so far the "identical twin" case, in which observations are supposed to be exactly compatible with one model solution. This is not so in reality because of observing and modelling errors. We shall now extend the formalism of Section 2 to the case when such observing and/or modelling errors are present.

First, one particular solution of the model (which need not be defined more precisely at this stage) is chosen as a reference solution. Then, for each observed datum, an "error" is defined as the difference between the datum itself and the corresponding value for the reference solution just chosen. The vector made up of all the errors thus defined will be noted E. With the notation of Section 2, the dimension of E is equal to the product Np of the number N of observation times by the dimension p of the observed vector X.

Denoting again by ΔY_n the Y-difference between the model state and the reference solution at the end of the n-th assimilation cycle, a derivation similar to that of Section 2 leads to the following relationship between ΔY_n and ΔY_{n+1}

$$\Delta Y_{n+1} = A\, \Delta Y_n + BE + o(\Delta Y_n, E) \tag{32}$$

where A and B are matrices with respective dimensions q x q and q x Np. A is the same matrix as in (9) since (32) must reduce to (9) when E=0. The matrix B, like the matrix A, is entirely determined by the resolvent matrix of the linearised perturbation system (3) in the vicinity of the reference solution.

Now the question is: as the number n of assimilation cycle increases to infinity, and E remains constant, how will ΔY_n behave? The following result is proved in T77: if the spectral radius of A is strictly less than 1, and if E and ΔY_o are small enough, ΔY_n will tend to a limit as n will tend to infinity. However, this limit will not in general be 0.

This result means that, if the observed values remain close enough to one particular model solution, and if the spectral radius of the amplification matrix corresponding to that solution is strictly less than 1, then the assimilation will converge to a limit. However, the corresponding model states at times $t_1, t_2 \ldots t_N$ will not in general lie on one model solution.

The proof given in T77 is too general to provide information of specific interest for the meteorological problem. It does not in particular allow any practical estimate of how "small" E must be for an assimilation to remain convergent. It is probably mostly through numerical experimentation that precise information can be obtained on this point.

6. A THEORETICAL WAY FOR ACCELERATING CONVERGENCE

We assumed in Section 2 that one particular arbitrary choice had been made as for the nature of the vector Y which was kept unchanged when observations of X were introduced. One can wonder how the convergence properties of an assimilation will be modified if it is another vector W, and not Y, which is kept unchanged when Y is updated. It is not difficult to show (Talagrand, 1980) that keeping W unchanged is equivalent to adding to Y the following correction whenever X is updated

$$\Delta'Y = D\Delta X + o(\Delta X,\Delta Y) \qquad (33)$$

In this expression D is a q x p matrix, obtained from the dependence of W with respect to X and Y, while ΔX and ΔY are, as before, the X and Y differences between the model and the observed solution. Conversely, any correction of type (33) applied to Y when X is updated keeps unchanged some perfectly defined vector W. We will consider in this section how the convergence properties of an assimilation are modified by a correction of type (33). In fact, only the matrix D will be important for this study and it will not be necessary to consider the corresponding "invariant" vector W.

Many assimilation procedures have been defined, some of which are in operational use, which use a correction of type (33). For a large number of them (but not all) the correction is purely linear, i.e. there is no $o(\Delta X,\Delta Y)$ term. The "optimal analysis" when performed with the model forecast as first guess, as is most often the case (see Lorenc et al. (1977)), is one of them. The matrix D, which is then determined on the basis of statistical considerations is in that case equal to $-QC^{-1}$, where Q is the covariance matrix of ΔY with ΔX, and C the variance-covariance matrix of ΔX with itself. Various schemes, intended at restoring the geostrophic balance disrupted by the introduction of observations, and which are all particular cases of (33), have been defined by Kistler and McPherson (1975) and by Daley and Puri (1980). Several of them have been shown to accelerate the convergence rate of an assimilation. For still another example of (33), see Tadjbakhsh (1969).

The scheme proposed by Kistler and McPherson is of particular interest because its simplicity allows for a ready analysis of its effects. This scheme consists, whenever the mass field is updated, in adding on the wind field a correction which is geostrophically related to the mass correction. The corresponding matrix D is the matrix which expresses the dependence of geostrophic wind on the mass field. Since the global correction thus added on the flow is geostrophic, the gravity wave component of the difference flow is not modified at an introduction time. In a linear model, this component is not modified either between two integrations. The Kistler-McPherson correction used alone in a linear model therefore prevents the convergence of an assimilation, since it maintains the gravity wave component of the difference flow. The situation becomes completely different however if the observed solution is known to be exactly geostrophic and the Kistler-McPherson correction is used together with a geostrophic adjustment procedure. After geostrophic adjustment has been applied once, the difference flow is and remains geostrophic. The energy convervation argument presented in Section 2 then applies in the particular case of a purely geostrphic flow, and an assimilation will converge under the only condition that the available observations uniquely define the observed geostrophic solution. In the case of a periodic f-plane, it is easy to show that the assimilation will have exactly converged (i.e. the difference flow will have been reduced to exactly 0) as soon as the mass field has been updated at least once at each grid point.

Coming back to the general correction scheme (33), we are now going to study how such a scheme modifies the amplification matrix (10). We will assume that the correction matrix D can vary with the introduction time. After introduction of observations of X, and correction of Y, at time t_i, we are left with differences $\Delta X = 0$ and $\Delta Y(t_i)$. Integrating the model to the next introduction time t_{i+1} will produce the following differences (Eqs. (6)).

$$\Delta_1 X(t_{i+1}) = R_x^Y (t_{i+1}, t_i) \Delta Y(t_i) + o(\Delta Y(t_i))$$

$$\Delta_1 Y(t_{i+1}) = R_Y^Y (t_{i+1}, t_i) \Delta Y(t_i) + o(\Delta Y(t_i))$$

Introducing the observations at time t_{i+1}, and correcting Y according to (33) will set ΔX to 0 and transform ΔY into

$$\Delta Y(t_{i+1}) = \Delta_1 Y(t_{i+1}) + D(t_{i+1}) \Delta_1 X(t_{i+1}) + o(\Delta_1 Y(t_{i+1}), \Delta_1 X(t_{i+1}))$$

$$= [R_Y^Y(t_{i+1}, t_i) + D(t_{i+1}) R_x^Y(t_{i+1}, t_i)] \Delta Y(t_{i+1}) + o(\Delta Y(t_i)) \quad (34)$$

The developments of Section 2 therefore remain valid, the matrix $R_Y^Y(t_{i+1}, t_i)$ being replaced at each assimilation step by

$$P_Y^Y(t_{i+1}, t_i) = R_Y^Y(t_{i+1}, t_i) + D(t_{i+1}) R_x^Y(t_{i+1}, t_i)$$

In particular the amplification matrix (10) now becomes

$$A' = P_Y^Y(\tau_1, \tau_M) P_Y^Y(\tau_M, \tau_{M-1}) \ldots\ldots\ldots P_Y^Y(\tau_2, \tau_1) \quad (35)$$

A correction of type (33) will be useful if the resulting spectral radius $\rho(A')$ is smaller than $\rho(A)$. A particularly interesting case is when the matrix A' can be made equal to 0 by an appropriate choice of the correction matrices $D(t_i)$. If this can be achieved, the decrease of the difference ΔY with the number of assimilation cycles will be faster than exponential. The following theorem is proved in T77.

Theorem (T) : The matrix A' can be made equal to 0 by an appropriate choice of the correction matrices $D(t_i)$ if, and only if, the following condition (C) is satisfied

(C): the only solution of the linearised perturbation system (3) which satisfies the condition $\delta X(t_i) = 0$ at all observation times t_i $(i=1,\ldots,N)$ is the null solution $\delta X(t) \equiv \delta Y(t) \equiv 0$.

Condition (C) essentially means that, in the approximation defined by the linearised system (3), the available observations $X(t_i)$ uniquely define the observed solution.

The proof of theorem (T) resorts only to basic notions of linear algebra. It turns out that a complete assimilation cycle is not necessary to make the matrix A' equal to 0, but that the product of the matrices P_Y^Y over either of the two phases (forward or backward) of a cycle can be made null by an appropriate choice of the correction matrices $D(t_i)$. Also, it is not necessary for theorem (T) to hold that the same parameters be observed at the successive observation times, but the nature and even the numbers of the observed parameters can vary with time. Moreover, the set of correction matrices which make A' equal to zero is in general not unique.

Example

Let us consider the linearised shallow-water equations on an f-plane (Eqs. (15)). For successive observations of the mass field ϕ separated by $\Delta\tau$, the matrices R_X^Y and R_Y^Y are obtained from the resolvent matrix (16), viz

$$R_X^Y = -\frac{\Phi_o}{\alpha}\begin{pmatrix}\sin\beta & \gamma(1-\cos\beta)\end{pmatrix}$$

$$R_Y^Y = \begin{pmatrix}\cos\beta & \gamma\sin\beta \\ -\gamma\sin\beta & 1-\gamma^2(1-\cos\beta)\end{pmatrix}$$

Given N successive observations of ϕ separated by $\Delta\tau$, we have seen in Section 2 that these observations uniquely define the corresponding solution of (15), i.e. they satisfy the above condition (C) if, and only if, the following conditions are simultaneously verified

$N > 3$

$\beta \neq \ell\pi$ $\quad \ell$ integer

$\gamma \neq 0$

$\gamma \neq 1$

Theorem (T) tells us that, under these same conditions, there exist correction matrices D, with dimensions 2 x 1, which make the matrix A' of (35) equal to 0. Since condition (C) requires only $N \geqslant 3$, two such correction matrices must be sufficient. There must therefore exist two 2 x 1 matrices D_1 and D_2 such that

$$(R_y^y + D_2 R_x^y)(R_y^y + D_1 R_x^y) = 0$$

An easy calculation shows that one solution for this equation is

$$D_1 = D_2 = \frac{\alpha}{2\Phi_o} \begin{pmatrix} \dfrac{1 + 2\cos\beta}{\sin\beta} \\ \\ \dfrac{1-2\gamma^2(1-\cos\beta)}{\gamma(1-\cos\beta)} \end{pmatrix}$$

This is always defined, except for values of Φ_o, β, γ for which condition (C) is not satisfied anyway.

Theorem (T) provides a theoretical basis for optimising the convergence of an assimilation. However, the explicit computation of a set of "optimal" correction matrices $D(t_i)$ in an operational assimilation raises a number of difficulties, the most basic of which is the following: the optimal matrices depend in the linearised perturbation system (3). The latter, in the case when the basic equations (1) are non-linear, depends in turn on the observed solution, which is precisely what is being looked for. It is therefore certainly impossible to determine the correction matrices which make the amplification matrix A' exactly equal to 0. But, since an assimilation performed with meteorological equations already converges with no correction of type (33) at all, it is reasonable to assume that its convergence can be accelerated by optimal matrices corresponding, not to the solution which is actually observed, but to some already known solution which can be considered as being some approximation of the observed solution.

Finally, it must be added that theorem (T) *per se* provides a way of using only what can be called the *dynamical* information, obtained from the fact that the solution to be determined must satisfy the basic equations (1). It is of no direct help for using any kind of *statistical* information, obtained from known statistical properties of the solution to be reconstituted. Indeed such statistical information is commonly used in present assimilation procedures under the form, for instance, of structure functions. Using statistical information amounts to reducing the number of independent degrees of freedom to be determined. A fully efficient assimilation procedure must use information of both kinds. A possible way for doing so could be to restrict the linearised perturbation system (3) to those modes which are compatible with the *a priori* imposed statistical constraints.

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