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# ON BALANCE REQUIREMENTS AS INITIAL CONDITIONS

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## Abstract

The problem of initialisation is considered. The study is based on the shallow water equations permitting gravity and Rossby waves. Perturbations on a state of rest are treated as a linear problem, and the perturbations are permitted to vary in two horizontal dimensions.

The geometry is a plane on which the Coriolis parameter varies linearly with the meridional coordinate. This means that the treatment is different from the standard beta-plane. The eigen-value problem is solved and the eigen-functions are Hermite polynominals multiplied by an exponential function. The set of eigen-functions is orthogonal over the infinite plane.

The solutions are fitted to initial conditions, and it is possible to find the partitioning of the initial amplitudes on the various wave types. It is thus possible to make a comparative study between different initialisation procedures including non-divergent and quasi-non-divergent initial conditions. Normal mode initialisation is also considered. It is demonstrated that the former procedures result in significant amplitudes in the gravity waves on the large scale. The relations between the wind field and the mass field are derived for the normal mode initialisation.

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## 1. Introduction

The initial conditions to be used in numerical weather forecasting with the primitive equations are difficult to specify. It appears desirable to determine the conditions in such a way that the amplitude of the gravity waves is small, which means that some form of balance will have to exist between the mass field and the wind field. It has therefore been customary to impose such a balance through one or the other form of the so-called balance equation.

Hinkelmann's classical study (1951) of the meteorological noise problem indicated that an initial balance will reduce the amplitude of the gravity waves considerably as compared to those which would be present without the required balance conditions in the initial state. The investigations by Hinkelmann (loc.cit.) as well as later studies by Phillips (1960) and Gollvik and Thaning (1977) have assumed that the perturbations depended on the zonal, vertical and time colordinates, but not on the meridional colordinate. Studies by Wiin-Nielsen (1971)and Kasahara (1976) have used a spherical geometry, but the determination of the structure and the speed of the waves become rather cumbersome in this case because the frequency equation must be solved in this case by expanding the variables in infinite series of either associated Legendre functions or Hough functions.

One of the most promising developments in initialisation procedures in recent years is the so-called normal mode initialisation. This topic has been treated extensively in the literature. A recent paper by Daley (1978) summarises the most important developments including the non linear normal mode initialisation by Machenhauer (1977). In spite of the extensive investigations it is nevertheless of interest to compare various initialisation schemes using a simple model and to obtain from the model an explicit solution which, from the linear point of view, eliminates the gravity waves completely. Such a model will be presented in this paper.

In the present study we shall adopt the shallow water equations. We shall formulate the equations of the problem on a plane on which it will be assumed that the Coriolis parameter varies linearly with the south-north co-ordinate. The variations will be periodic in the east-west direction, and the plane will be infinite.

The assumptions made here are thus different from the usual beta plane assumptions. We shall furthermore require that the perturbation quantities remain finite at infinity in the south and north directions.

This formulation has the advantage that it incorporates the required space dimensions, that the frequency equation is relatively simple, and that the structure of the waves consequently can be evaluated rather easily. Although the present paper will treat the linear case only it is possible to expand the investigation to include nonlinear effects.

## 2. The Model

The basic equations are the standard equations for a shallow water model. The co-ordinates x, y and t are replaced by nondimensional quantities $\xi$, $\eta$ and $\tau$ as follows

$$x = \frac{fo}{\beta} \xi; \; y = \frac{fo}{\beta} (\eta - 1); \; t = \frac{1}{fo} \tau \qquad (2.1)$$

while the velocity components and the geopotential are scaled in the following way

$$u = \frac{\beta}{fo^2} \Phi_o U; \; v = \frac{\beta}{fo^2} \Phi_o V; \; \phi = \Phi_o \Phi \qquad (2.2)$$

With this scaling the equations become

$$\frac{\partial U}{\partial \tau} + q^2 \left[ U \frac{\partial U}{\partial \xi} + V \frac{\partial U}{\partial \eta} \right] = - \frac{\partial \Phi}{\partial \xi} + \eta V$$

$$\frac{\partial V}{\partial \tau} + q^2 \left[ U \frac{\partial V}{\partial \xi} + V \frac{\partial V}{\partial \eta} \right] = - \frac{\partial \Phi}{\partial \eta} - \eta U \qquad (2.3)$$

$$\frac{\partial \Phi}{\partial \tau} + q^2 \left[ U \frac{\partial \Phi}{\partial \xi} + V \frac{\partial \Phi}{\partial \eta} \right] = - q^2 \Phi \left[ \frac{\partial U}{\partial \xi} + \frac{\partial V}{\partial \eta} \right]$$

with

$$q = \frac{\beta}{f_o^2} \sqrt{\Phi_o} \qquad (2.4)$$

Using the values $\beta = 16 \times 10^{-12} m^{-1} s^{-1}$, $f_o = 10^{-4} s^{-1}$ and $\Phi_o = 9.8 \times 8800\ m^2 s^{-2}$ we find $q = 0.47$.

It turns out to be convenient to express each of the dependent variables in a series of the form

$$z(\xi,\eta,\tau) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} z(m,\eta,\tau)\, \psi_n(\eta)\, e^{imk\xi} \qquad (2.5)$$

where $k = 2\pi/L$ and L is the zonal length under consideration. $\psi_n(\eta)$, related to the Hermite polynomials, and the properties of the class $\psi_n(\eta)$ are given in Appendix 1. The form (2.5) is selected because these functions naturally appear in the linearised perturbation problem.

The linearised equations are obtained from (2.3) by neglecting the terms in the brackets on the left hand side of the equations and by setting $\Phi = 1$ on the right hand side of the last equations.

Using (2.3) and (2.5) together with the orthogonality properties of the functions $\exp(imk\xi)$ and $\psi_n(\eta)$ we may write the system (2.3) in the form :

$$\frac{dU(m,n)}{d\tau} = - q^2 \sum_p \sum_{n_1} \sum_{n_2} ipk\, U(m-p,n_1)\, U(p,n_2)\, I(n_1,n_2,n)$$

$$- q^2 \sum_p \sum_{n_1} \sum_{n_2} \tfrac{1}{2} V(m-p,n_1)\, U(p,n_2) \{ \sqrt{n_2}\, I(n_1,n_2-1,n) -$$

$$- \sqrt{n_2+1}\, I(n_1,n_2+1,n)\}$$

$$- imk\, \Phi(m,n) + \sqrt{n}\, V(m,n-1) + \sqrt{n+1}\, V(m,n+1)$$

$$\frac{dV(m,n)}{d\tau} = - q^2 \sum_p \sum_{n_1} \sum_{n_2} ipk\ U(m-p,n_1)\ V(p,n_2)\ I\ (n,n_1,n_2)$$

$$- q^2 \sum_p \sum_{n_1} \sum_{n_2} \tfrac{1}{2}\ V(m-p,n_1)\ V(p,n_2)\ \{\sqrt{n_2}\ I(n_1,n_2-1,n) -$$

$$- \sqrt{n_2+1}\ I(n,n_2+1,n)\}$$

$$-\tfrac{1}{2}\sqrt{n+1}\ \Phi\ (m,n+1) + \tfrac{1}{2}\sqrt{n}\ \Phi\ (m,n-1) \qquad (2.6)$$

$$- \sqrt{n}\ U(m,n-1) - \sqrt{n+1}\ U(m,n+1)$$

$$\frac{d\Phi(m,n)}{d\tau} = - q^2 \sum_p \sum_{n_1} \sum_{n_2} ipk\ U(m-p,n_1)\ \Phi\ (p,n_2)\ I(n_1,n_2,n)$$

$$- q^2 \sum_p \sum_{n_1} \sum_{n_2} \tfrac{1}{2}\ V(m-p,n_1)\ \Phi(p,n_2)\{\sqrt{n_2}\ I(n_1,n_2-1,n) -$$

$$- \sqrt{n_2+1}\ I(n_1,n_2+1,n)\}$$

$$- q^2 \sum_p \sum_{n_1} \sum_{n_2} ipk\ \Phi(m-p,n_1)\ U(p,n_2)\ I(n_1,n_2,n)$$

$$- q^2 \sum_p \sum_{n_1} \sum_{n_2} \tfrac{1}{2}\ \Phi(m-p,n_1)\ V(p,n_2)\{\sqrt{n_2}\ I(n_1,n_2-1,n) -$$

$$- \sqrt{n_2+1}\ I(n_1,n_2+1,n)\}$$

where

$$I(n_1,n_2,n) = \int_{-\infty}^{\infty} \psi_{n_1}(\eta)\ \psi_{n_2}(\eta)\ \psi\ (\eta)d\eta \qquad (2.7)$$

The system (2.6) can be used for numerical integrations. We note that the variables U(m,n), V(m,n) and $\Phi$ (m,n) are connected through the linear terms, i.e. the pressure force and the Coriolis force, in such a way that U(m,n) communicates with $\Phi$(m,n) , V(m,n-1) and V(m,n+1), while V(m,n-1) communicates with U(m,n-1), U(m,n+1), $\Phi$(m,n-1) and $\Phi$(m,n+1).

With respect to n we may say that even (odd) components in U and $\Phi$ communicate with odd (even) components in V. These properties make it possible to form low order systems from the general equations (2.6). Such systems will be considered later in the paper.

The integrals (2.7) represent interaction coefficients. They can normally be calculated exactly as demonstrated in Appendix 2.

## 3. The linear problem

The differential equations for this problem are obtained from (2.3) and take the form

$$\frac{\partial U}{\partial \tau} = - \frac{\partial \Phi}{\partial \xi} + \eta V$$

$$\frac{\partial V}{\partial \tau} = - \frac{\partial \Phi}{\partial \eta} - \eta U \qquad (3.1)$$

$$\frac{\partial \Phi}{\partial \tau} = - q^2 \left( \frac{\partial U}{\partial \xi} + \frac{\partial V}{\partial \eta} \right)$$

We shall first solve these equations using perturbations of the type

$$\begin{Bmatrix} U \\ V \\ \Phi \end{Bmatrix} = \begin{Bmatrix} \hat{U}(\eta) \\ \hat{V}(\eta) \\ \hat{\Phi}(\eta) \end{Bmatrix} e^{ik(\xi - C\tau)} \qquad (3.2)$$

(3.2) is inserted in (3.1). We express $\hat{U}$ and $\hat{\Phi}$ in terms of $\hat{V}$ and obtain

$$\hat{U} = \frac{ik}{k^2(q^2 - C^2)} \left[ q^2 \frac{d\hat{V}}{d\eta} - C\eta\hat{V} \right] \qquad (3.3)$$

$$\hat{\Phi} = \frac{ikq^2}{k^2(q^2 - C^2)} \left[ C \frac{d\hat{V}}{d\eta} - \eta\hat{V} \right]$$

Using the expressions (3.3) it is easy to derive a single equation for $\hat{V}$. This differential equation is :

$$\frac{d^2\hat{V}}{d\eta^2} + \left\{ - \frac{1}{q^2} \eta^2 - \frac{1}{C} - \frac{k^2}{q^2} (q^2 - C^2) \right\} \hat{V} = 0 \qquad (3.4)$$

(3.4) represents an eigenvalue problem and is a standard equation which is solvable in terms of Hermite polynomials and exponential functions as seen from Abramovitz and Stegun (1965) provided C satisfies the equation

$$- q \left[ \frac{1}{C} + \frac{k^2}{q^2} (q^2 - C^2) \right] = 2n+1 \qquad (3.5)$$

or

$$C^3 - \left[ (2n+1) \frac{q}{k^2} + q^2 \right] C - \frac{q^2}{k^2} = 0 \qquad (3.6)$$

(3.6) will have three roots corresponding to a Rossby wave (a small negative value of C) and to external gravity waves travelling in the positive and negative x-directions (large positive and negative values of C). It may be instructive to consider the dimensional form of (3.6). We note first that

$$C_* = \frac{f_o^2}{\beta} C, \quad k_* = \frac{\beta}{f_o} k \qquad (3.7)$$

where the asterisk denotes the dimensional quantities. Noting further that

$$q = \frac{C_R C_g}{C_I^2} ; \quad C_R = \frac{\beta}{k_*^2}, \quad C_I = \frac{f_o}{k_*}, \quad C_g = \sqrt{\Phi_o} \qquad (3.8)$$

we find from (3.6) that $c_*$ satisfies the equation

$$c_*^3 - [(2n+1)\,c_R c_g + c_g^2]\,c_* - c_R c_g^2 = 0 \qquad (3.9).$$

From (3.9) it can be seen that an approximative value of the speed of the Rossby wave is

$$c_{*1} = - \frac{c_R}{1 + (2n+1)\frac{c_R}{c_g}} \qquad (3.10)$$

while the corresponding approximate values for the gravity waves are

$$c_{*\,2,3} = \pm c_g \sqrt{1 + (2n+1)\frac{c_R}{c_g}} \qquad (3.11)$$

When c satisfies (3.5) we may find the expressions for $\hat{U}$, $\hat{V}$ and $\hat{\phi}$. To get convenient expressions it is an advantage to introduce a variable $\eta_*$ by the expression

$$\eta = \sqrt{\frac{q}{2}} \cdot \eta_* \qquad (3.12).$$

Using furthermore (3.5) we find that (3.4) is written in the form

$$\frac{d^2\hat{V}}{d\eta_*^2} + \left\{ -\frac{1}{4}\eta_*^2 + \frac{2n+1}{2} \right\} \hat{V} = 0 \qquad (3.13).$$

Writing further

$$\hat{V} = e^{-\frac{1}{4}\eta_*^2}\,\hat{V}_* \qquad (3.14)$$

we find that $\hat{V}_*$ satisfies the equation

$$\frac{d^2\hat{V}_*}{d\eta_*^2} - \eta_* \frac{d\hat{V}_*}{d\eta_*} + n\,\hat{V}_* = 0 \qquad (3.15)$$

which is the differential equation for the Hermite polynomial $He_n(\eta_*)$ . It follows therefore that the solution to (3.4) can be written in the form

$$\hat{V}(\eta_*) = V_n \, e^{-\frac{1}{4}\eta_*^2} \, He_n(\eta_*) = V_n \psi_n(\eta_*) \qquad (3.16)$$

where $V_n$ is redefined in such a way that $\psi_n$ is normalised as defined in Appendix 1. Solutions for $\hat{U}$ and $\hat{\Phi}$ are found by introducing the variable $\eta_*$ in (3.3), inserting from (3.16) and using the relations given in Appendix 1 for the functions $\psi_n$ . We get :

$$\hat{U}(\eta_*) = \frac{ik}{k^2}\sqrt{\frac{nq}{2}}\,\frac{1}{q+C}\,V_n\,\psi_{n-1} - \frac{ik}{k^2}\sqrt{\frac{(n+1)q}{2}}\,\frac{1}{q-C}\,V_n\,\psi_{n+1}$$

$$\hat{\Phi}(\eta_*) = -\frac{ik}{k^2}\,q\sqrt{\frac{nq}{2}}\,\frac{1}{q+C}\,V_n\,\psi_{n-1} - \frac{ik}{k^2}\,q\sqrt{\frac{(n+1)q}{2}}\,\frac{1}{q-C}\,V_n\,\psi_{n+1}$$

$$(3.17)$$

(3.17) shows as expected that if $\hat{V}$ is even (odd) $\hat{U}$ and will be odd (even). The constants $V_n$ are determined by the initial conditions as follows. We note that $\hat{U}$ and $\hat{\Phi}$ have the form

$$\hat{U} = U_1\,\psi_{n-1} + U_2\,\psi_{n+1}$$

$$\hat{\Phi} = \Phi_1\,\psi_{n-1} + \Phi_2\,\psi_{n+1} \qquad (3.18)$$

The initial conditions must be specified in this form also. Denoting the initial values by $\hat{V}_0$, $\hat{U}_{10}$ , $\hat{U}_{20}$ , $\hat{\Phi}_{10}$ and $\hat{\Phi}_{20}$ and noting that three different modes exist corresponding to the three values of the phase speed ($C_1$, $C_2$ and $C_3$) we have the following equations

$$
\begin{aligned}
\hat{V}_1 + \hat{V}_2 + \hat{V}_3 &= \hat{V}_o \\
\hat{U}_{11} + \hat{U}_{12} + \hat{U}_{13} &= \hat{U}_{10} \\
\hat{U}_{21} + \hat{U}_{22} + \hat{U}_{23} &= \hat{U}_{20} \\
\hat{\Phi}_{11} + \hat{\Phi}_{12} + \hat{\Phi}_{13} &= \hat{\Phi}_{10} \\
\hat{\Phi}_{21} + \hat{\Phi}_{22} + \hat{\Phi}_{23} &= \hat{\Phi}_{20}
\end{aligned}
\qquad (3.19)
$$

To solve these equations we note from (3.17) that

$$
\begin{aligned}
\hat{\Phi}_{11} &= -q\,\hat{U}_{11}, & \hat{\Phi}_{12} &= -q\,\hat{U}_{12}, & \hat{\Phi}_{13} &= -q\,\hat{U}_{13} \\
\hat{\Phi}_{21} &= q\,\hat{U}_{21}, & \hat{\Phi}_{22} &= q\,\hat{U}_{22}, & \hat{\Phi}_{23} &= q\,\hat{U}_{23}
\end{aligned}
\qquad (3.20)
$$

It follows therefore that (3.19) will have non-trivial solutions only if

$$
\hat{\Phi}_{10} = -q\,\hat{U}_{10}\ , \quad \hat{\Phi}_{20} = q\,\hat{U}_{20} \qquad (3.21).
$$

To obtain solutions to our system it is thus necessary to further restrict the initial conditions in such a way that (3.21) is satisfied. As we shall see later (3.21) is quite restrictive. Inserting from (3.17) in (3.19) disregarding the last two equations because of (3.21) we obtain:

$$
\begin{aligned}
&\hat{V}_1 + \hat{V}_2 + \hat{V}_3 = \hat{V}_0 \\
&\frac{ik}{k^2}\sqrt{\frac{nq}{2}}\,\frac{1}{q+C_1}\hat{V}_1 + \frac{ik}{k^2}\sqrt{\frac{nq}{2}}\,\frac{1}{q+C_2}\hat{V}_2 + \frac{ik}{k^2}\sqrt{\frac{nq}{2}}\,\frac{1}{q+C_3}\hat{V}_3 = \hat{U}_{10} \\
&-\frac{ik}{k^2}\sqrt{\frac{(n+1)q}{2}}\,\frac{1}{q-C_1}\hat{V}_1 - \frac{ik}{k^2}\sqrt{\frac{(n+1)q}{2}}\,\frac{1}{q-C_2}\hat{V}_2 + \frac{ik}{k^2}\sqrt{\frac{(n+1)q}{2}}\,\frac{1}{q-C_3}\hat{V}_3 = \hat{U}_{20}
\end{aligned}
\qquad (3.22)
$$

The determinant of (3.22) is

$$\Delta = \frac{1}{4} \frac{k^2}{q^2} \frac{1}{\sqrt{n(n+1)}} \left[ C_1 C_2 (C_1 - C_2) + C_2 C_3 (C_2 - C_3) + C_3 C_1 (C_3 - C_1) \right] \qquad (3.23)$$

and the solutions for $\hat{V}_1$, $\hat{V}_2$ and $\hat{V}_3$ are :

$$\begin{aligned}
\hat{V}_1 = & - \frac{1}{\Delta} \frac{q^2}{k^2} \sqrt{n(n+1)} \frac{C_2 - C_3}{(q^2 - C_2^2)(q^2 - C_3^2)} \hat{V}_o - \frac{1}{\Delta} \frac{ik}{k^2} \\
& \sqrt{\frac{(n+1)q}{2}} \frac{C_2 - C_3}{(q - C_2)(q - C_3)} \hat{U}_{10} \\
& + \frac{1}{\Delta} \frac{ik}{k^2} \sqrt{\frac{nq}{2}} \frac{C_2 - C_3}{(q + C_2)(q + C_3)} \hat{U}_{20} \\
\hat{V}_2 = & - \frac{1}{\Delta} \frac{q^2}{k^2} \sqrt{n(n+1)} \frac{C_3 - C_1}{(q^2 - C_3^2)(q^2 - C_1^2)} \hat{V}_o - \frac{1}{\Delta} \frac{ik}{k^2} \qquad (3.24) \\
& \sqrt{\frac{(n+1)q}{2}} \frac{C_3 - C_1}{(q - C_3))q - C_1)} \hat{U}_{10} \\
& + \frac{1}{\Delta} \frac{ik}{k^2} \sqrt{\frac{nq}{2}} \frac{C_3 - C_1}{(q + C_3)(q + C_1)} \hat{U}_{20} \\
\hat{V}_3 = & - \frac{1}{\Delta} \frac{q^2}{k^2} \sqrt{n(n+1)} \frac{C_1 - C_2}{(q^2 - C_1^2)(q^2 - C_2^2)} \hat{V}_o - \frac{1}{\Delta} \frac{ik}{k^2} \\
& \sqrt{\frac{(n+1)q}{2}} \frac{C_1 - C_2}{(q - C_1)(q - C_2)} \hat{U}_{10} \\
& + \frac{1}{\Delta} \frac{ik}{k^2} \sqrt{\frac{nq}{2}} \frac{C_1 - C_2}{(q + C_1)(q + C_2)} \hat{U}_{20}
\end{aligned}$$

where the expressions for $\hat{V}_2$ and $\hat{V}_3$ are obtained by a cyclic permutation of $C_1$, $C_2$ and $C_3$.

We note that the determinant (3.23) is independent of the specific values of the roots and depends on the coefficients of the cubic equation only. It can be shown that

$$\Delta = -\frac{1}{4}\frac{k^2}{q^2}\frac{1}{\sqrt{n(n+1)}}\sqrt{4\left[(2n+1)\frac{q}{k}+q^2\right]^3-27\frac{q^4}{k^4}} \quad (3.25)$$

The expressions derived in (3.24) and (3.25) are general and can be used to calculate $\hat{V}_1$ , $\hat{V}_2$ and $\hat{V}_3$ for any permissible initial conditions. The corresponding values of the zonal wind components are obtained from (3.17) and those of the geopotential from (3.21).

As mentioned in the introduction it is required to specify the initial conditions in such a way that the amplitudes of the gravity waves vanish. This condition can obviously be satisfied exactly in our linear model by requiring that $\hat{V}_2 = \hat{V}_3 = 0$ in (3.24). These two relations result in two inhomogeneous linear equations. We select to express $\hat{U}_{10}$ and $\hat{U}_{20}$ in terms of $\hat{V}_0$ and obtain

$$\begin{aligned}\hat{U}_{10} &= \frac{ik}{k^2}\sqrt{\frac{nq}{2}}\;\frac{1}{q+C_1}\;\hat{V}\\ \hat{U}_{20} &= -\frac{ik}{k^2}\sqrt{\frac{(n+1)q}{2}}\;\frac{1}{q-C_1}\;\hat{V}_o\end{aligned} \quad (3.26)$$

Since (3.21) must be satisfied we may rewrite (3.26) in the form

$$\begin{aligned}\hat{V}_o &= ik\sqrt{\frac{}{nq}}\left(1+\frac{C_1}{q}\right)\hat{\Phi}_{10}\\ \hat{V}_o &= ik\sqrt{\frac{2}{(n+1)q}}\left(1-\frac{C_1}{q}\right)\hat{\Phi}_{20}\end{aligned} \quad (3.27)$$

If (3.26) and (3.21) are satisfied we will have vanishing gravity waves during an integration of the linear model. These equations correspond therefore to the normal mode initialisation. It is of interest to compare these relations for the ideal initial conditions with other initial conditions which have been used. The most simple starting condition is the geostrophic relation.

Considering first the meridional component we note from (3.1) that we would impose the condition

$$\eta V_g = \frac{\partial \Phi}{\partial \xi} \tag{3.28}$$

Considering (3.2) and (3.12) we may write (3.28) in the form

$$\sqrt{\frac{q}{2}}\, \eta_* \hat{V}_g = ik\, \Phi \tag{3.29}$$

Using (3.18) and (A1.7) from Appendix 1 we find that

$$\begin{aligned} \hat{V}_{go} &= ik\sqrt{\frac{2}{nq}}\, \hat{\Phi} \\ \hat{V}_{go} &= ik\sqrt{\frac{2}{(n+1)q}}\, \hat{\Phi} \end{aligned} \tag{3.30}$$

It is thus seen from (3.27) and (3.30) that the geostrophic relation for the meridional wind component will result in small amplitudes of the gravity waves <u>only</u> when $C_1 \quad q$ which will be the case for sufficiently small scales.

We are next going to show that the full geostrophic relation is not a permissible initial condition for our system because of (3.21). Introducing (3.21) in (3.30) we find that

$$\hat{U}_{10} = -\frac{1}{ik}\sqrt{\frac{n}{2q}}\, \hat{V}_{go} \;; \qquad \hat{U}_{20} = \frac{1}{ik}\sqrt{\frac{n+1}{2q}}\, \hat{V}_{go} \tag{3.31}$$

The proof will be completed if we can demonstrate that (3.31) is in conflixt with the zonal geostrophic relation determined from

$$\eta \hat{U}_o + \frac{d\, \Phi_o}{d\eta} = 0 \tag{3.32}$$

Using again (3.18) and (A1.7) we find after some calculation that (3.32) reduces to the condition

$$\hat{U}_{10}\sqrt{n} + \hat{U}_{20}\sqrt{n+1} = 0 \qquad (3.33)$$

It is obvious that (3.33) is not satisfied by the components in (3.31). We have thus shown that the geostrophic assumption with a proper variation of the Coriolis parameter is non-permissible as an initial condition. It is however easy to show that (3.30) and (3.31) are the proper initial conditions if we require that the initial horizontal divergence shall vanish. We find

$$D = \left[ik\,\hat{U}_{10} + \sqrt{\frac{n}{2q}}\;\hat{V}_o\right]\psi_{n-1} + \left[ik\,\hat{U}_{20} - \sqrt{\frac{n+1}{2q}}\;\hat{V}_o\right]\psi_{n+1} \qquad (3.34).$$

If we require that D shall vanish we obtain (3.31). Considering therefore the case that we impose D = 0 as an initial condition we find from (3.24) after some calculations that

$$\frac{\hat{V}_1}{\hat{V}_o} = \frac{\sqrt{n(n+1)}}{\Delta \cdot k^2} \frac{C_2 C_3 (C_2 - C_3)}{(q^2 - C_2^2)(q^2 - C_3^2)}$$

$$\frac{\hat{V}_2}{\hat{V}_o} = \frac{\sqrt{n(n+1)}}{\Delta \cdot k^2} \frac{C_3 C_1 (C_3 - C_1)}{(q^2 - C_3^2)(q^2 - C_1^2)} \qquad (3.35)$$

$$\frac{\hat{V}_3}{V_o} = \frac{\sqrt{n(n+1)}}{\Delta \cdot k^2} \frac{C_1 C_2 (C_1 - C_2)}{(q^2 - C_1^2)(q^2 - C_2^2)}$$

From these expressions one may easily calculate the ratios of the components $\hat{U}_{11}$, $\hat{U}_{12}$, $\hat{U}_{13}$, $\hat{U}_{21}$, $\hat{U}_{22}$, and $\hat{U}_{23}$ to the initial values $\hat{U}_{10}$ and $\hat{U}_{20}$, respectively.

Since the only geostrophic initial state is one of rest it may also be of interest to consider the partitioning of initial amplitudes into Rossby waves and gravity waves for a couple of other cases. Assume for example that we have a non-geostrophic deviation in the zonal windfield or, equivalently, in the height field because of (3.21). We may then calculate how this initial amplitude in, say, $\hat{U}_{10}$ is distributed among $\hat{U}_{11}$, $\hat{U}_{12}$ and $\hat{U}_{13}$ and how much we find in the other zonal component and the meridional component. To be specific, if $\hat{U}_{10} \neq 0$, $\hat{U}_{20} = \hat{V}_o = 0$ we find that

$$\frac{\hat{V}_1}{\hat{U}_{10}} = -\frac{1}{}\frac{ik}{k^2}\sqrt{\frac{(n+1)\,q}{2}}\quad\frac{C_2 - C_3}{(q-C_2)(q-C_3)}$$

$$\frac{\hat{U}_{11}}{\hat{U}_{10}} = \frac{1}{}\frac{1}{k^2}\;\frac{q}{2}\;\sqrt{n(n+1)}\quad\frac{C_2 - C_3}{(q+C_1)(q-C_2)(q-C_3)} \tag{3.36}$$

$$\frac{\hat{U}_{21}}{\hat{U}_{10}} = -\frac{1}{\Delta}\frac{1}{k^2}\frac{q}{2}\,(n+1)\quad\frac{C_2 - C_3}{(q-C_1)(q-C_2)(q-C_3)}$$

where the remaining values are obtained by cyclic permutations among $C_1$ , $C_2$ and $C_3$.

Similarly, if the initial non-geostrophic amplitude is in $\hat{V}_o$, but not in the zonal wind and geopotential field we find for $\hat{V}_o \neq 0$, $\hat{U}_{10} = \hat{U}_{20} = 0$ that

$$\frac{\hat{V}_1}{\hat{V}_o} = -\frac{\sqrt{n(n+1)}}{\Delta}\;\frac{q^2}{k^2}\;\frac{C_2 - C_3}{(q^2-C_2^2)(q^2-C_3^2)}$$

$$\frac{\hat{U}_{11}}{\hat{V}_o} = -\frac{n\sqrt{n+1}}{\Delta\;\sqrt{2}}\;\frac{q^2\sqrt{q}}{k^4}\;ik\;\frac{C_2 - C_3}{(q+C_1)(q^2-C_2^2)(q^2-C_3^2)} \tag{3.37}$$

$$\frac{\hat{U}_{21}}{\hat{V}_o} = +\frac{(n+1)\sqrt{n}}{\Delta\;\sqrt{2}}\;\frac{q^2\sqrt{q}}{k^4}\;ik\;\frac{C_2 - C_3}{(q-C_1)(q^2-C_2^2)(q^2-C_3^2)}$$

## 4. Some numerical examples

This section contains some results of numerical calculations based on the analysis made in Section 3 of this paper.

The first example is based on the initial condition of no divergence, i.e. the equations (3.29) and (3.30). Figure 1, showing $V_1/V_o$ as a function of the zonal wave length measured in the unit $10^6$m and the meridional parameter n, indicates that $V_1/V_o$ is extremely small for all values of n as long as L, the zonal wave length, is small. However, as L increases the ratio $V_1/V_o$ will increase and for large values of L we see that the ratio increases with decreasing values of n and can become as large as about 0.2 for L = 28 and n = 1, i.e. for a large scale in both the zonal and meridional directions. A similar figure (not reproduced) is found for the ratio $V_2/V_o$. Figure 2 shows the ratio $V_3/V_o$ for the Rossby waves in the same arrangement. Here we observe that the ratio is very close to unity for all values of L and n. The details show that $V_3/V_o$ is slightly larger than unity for small values of L and less than one for large value of the zonal wave length. However, at no place is the value less than 0.965.

Fig. 3 shows the ratio $U_{12}/U_{10}$ which is in general small indicating that the initial assumption of no divergence reduces the amplitude of the gravity waves considerably. The exception is on the largest scale (L large, n small) where the ratio is about 0.2. Fig. 4 showing the ratio $U_{13}/U_{10}$ ,i.e. the amplitude of the Rossby waves, indicates that the ratio is everywhere larger than unity and may be as large as 1.3 for large L and small n. The ratios relating to $U_2$ , i.e. $U_{21}/U_{20}$, $U_{22}/U_{20}$ , $U_{23}/U_{20}$ , are similar to the corresponding ratios for $U_1$ except that $U_{23}/U_{20}$ is everywhere smaller than unity and is about 0.8 for large L and small n.

The general conclusion from this example is therefore that an initial condition of no divergence will be an effective filter of gravity type waves at small scales, but that the gravity waves contain a considerable fraction of the initial amplitude for the largest scales. Furthermore, the amplitude of the Rossby waves in the zonal motion, see Fig. 4, is considerably larger than the initial amplitude on the largest scale.

In the next example we shall assume that $U_{10} \neq 0$ while $U_{20} = V_0 = 0$. We may think of this case as a small disturbance introduced in one of the two zonal wind components. Under these circumstances we have

$$\begin{aligned} V_1 + V_2 + V_3 &= 0 \\ U_{11} + U_{12} + U_{13} &= U_{10} \\ U_{21} + U_{22} + U_{23} &= 0 \end{aligned} \qquad (4.1)$$

Fig. 5 shows that the response is a meridional wind field $V_2/U_{10}$ which is large (0.64) on the middle scales, but small when L is small. The response in the Rossby mode, $V_3/U_{10}$, shows a similar distribution as seen in Fig. 6. The gravity modes in the zonal wind, i.e. $U_{11}/U_{10}$ and $U_{12}/U_{10}$ , are very different because $U_{11}/U_{10}$ is relatively small for all values of L and n not exceeding about 0.15 anywhere (not shown). However, the other gravity mode $U_{12}/U_{10}$ has a large amplitude for small L decreasing to about 0.4 for large L (Fig. 7). The Rossby mode behaves in the opposite way with small values for small L increasing to about 0.5 for large L (Fig. 8). The fields $U_{21}/U_{10}$, $U_{22}/U_{10}$ and $U_{23}/U_{10}$ are such that the three ratios are small for small L. The two gravity modes increase in amplitude with L attaining maximum values of about 0.25 for large values of L. The corresponding Rossby mode is everywhere negative with a maximum absolute value of 0.5.

The last calculation assumes that $V_o \neq 0$ and $U_{10} = U_{20} = 0$. For this case we have

$$\begin{aligned} V_1 + V_2 + V_3 &= V_o \\ U_{11} + U_{12} + U_{13} &= 0 \\ U_{21} + U_{22} + U_{23} &= 0 \end{aligned} \qquad (4.2)$$

For the three modes of V we find that the two gravity modes, $V_1/V_o$ and $V_2/V_o$, are close to zero for small values of L but increase with L to maximum values of about 0.5. The Rossby mode is thus small for small L and decreases with L to quite small values at large L (Fig.9). The zonal gravity mode $U_{12}/V_o$ is shown in Fig. 10 with large negative values on the middle scales. The corresponding Rossby mode, $U_{13}/V_o$, in Fig. 11 is positive with maximum values on middle scales. Fig. 12 and Fig. 13 show finally $U_{21}/V_o$ and $U_{23}/V_o$, a gravity mode and the Rossby mode for this case.

In summary, for a given value of $V_o$ with $U_{10} = U_{20} = 0$ we find that the Rossby mode in the meridional wind is large only when the scale is small, while the Rossby mode in the zonal components is largest on a middle scale.

## 5. A Low-order System

The system considered in Section 3 is based on the primitive equations. Most initialisation schemes used in practice are based on the balance equation or, in more general cases, this equation combined with the quasi-geostrophic or quasi-balanced model equations which are used to calculate a balanced initial state, although the predictions themselves are done with the primitive equations. It is very difficult to use the system treated in Section 3 to test the various initialisation schemes because it is a minimal system.

In this section we consider a different minimal system which is derived from the vorticity, divergence and continuity equations. Our starting point is still (3.1). From the first two equations of motion we derive the vorticity and divergence equations in the usual manner. Denoting the stream function, the velocity potential and the geopotential by $S_*$, $\chi_*$ and $\Phi_*$, respectively, we find that

$$\frac{\partial \nabla^2 S_*}{\partial \tau} = -\eta \nabla^2 \chi_* - \frac{\partial S_*}{\partial \xi} - \frac{\partial \chi_*}{\partial \eta}$$

$$\frac{\partial \nabla^2 \chi_*}{\partial \tau} = -\nabla^2 \Phi_* + \eta \nabla^2 S_* + \frac{\partial S_*}{\partial \eta} - \frac{\partial \chi_*}{\partial \xi} \qquad (5.1)$$

$$\frac{\partial \Phi_*}{\partial \tau} = -q^2 \nabla^2 \chi_* \; .$$

We shall next assume that $S_*$, $\chi_*$ and $\Phi_*$ are developed in series of the form

$$\begin{Bmatrix} S_* \\ \chi_* \\ \Phi_* \end{Bmatrix} = \sum_m \sum_n \begin{Bmatrix} S(m,k,\tau) \\ \chi(m,k,\tau) \\ \Phi(m,k,\tau) \end{Bmatrix} \psi_n(\eta)\, e^{ikm\xi} \qquad (5.2)$$

The series (5.2) are substituted in the system (5.1). To reduce the resulting equations to a set of spectral equations it is necessary to find the series expansions for the types of terms appearing in (5.1). Using $S_*$ as an example it can be shown that the following series are valid :

$$\nabla^2 S_* = \sum_m \sum_n \{\tfrac{1}{4}\sqrt{(n-1)n}\, S(m,n-2) + \tfrac{1}{4}\sqrt{(n+1)(n+2)}\, S(m,n+2) - (\tfrac{1}{4}(2n+1) + k^2m^2)\, S(m,n)\}\, \psi_n(\eta)\, e^{imk\xi}$$

$$\eta\nabla^2 S_* = \sum_{mn}\sum \{\tfrac{1}{4}\sqrt{(n-2)(n-1)n}\, S(m,n-3) - \sqrt{n}(k^2m^2+\tfrac{1}{4}(n-2))\, S(m,n-1) - \sqrt{n+1}\,(k^2m^2+\tfrac{1}{4}(n+3))\, S(m,n+1) + \tfrac{1}{4}\sqrt{(n+1)(n+2)(n+3)}\, S(m,n+3)\}\psi_n\, e^{imk\xi} \qquad (5.3)$$

$$\frac{\partial S_*}{\partial \xi} = \sum_m \sum_n ikm\, S(m,n)\psi_n e^{imk\xi}$$

$$\frac{\partial S_*}{\partial \eta} = \sum_m \sum_n \left[\tfrac{1}{2}\sqrt{n+1}\, S(m,n+1) - \tfrac{1}{2}\sqrt{n}\, S(m,n-1)\right]\psi_n e^{imk\xi}$$

If we next make the decision that we shall consider a low order systen containing the components

$S(n-1)$, $\chi(n)$ and $\Phi(n)$

where all the components have the same value of m which, however, can be arbitrary, it is easy to derive the equations for such a system. To simplify the notations we introduce the symbols :

$$F_{-2} = m^2k^2 + \tfrac{1}{4}(n-2)$$
$$F_{-1} = m^2k^2 + \tfrac{1}{4}(2n-1)$$
$$F_o = m^2k^2 + \tfrac{1}{4}n \qquad (5.4)$$
$$F_1 = m^2k^2 + \tfrac{1}{4}(2n+1)$$
$$F_2 = m^2k^2 + \tfrac{1}{4}(n+2)$$

In the low order system we have then

$$\nabla^2 S_* = - F_{-1}\, S(n-1)\psi_n\, e^{imk\xi}$$

$$\eta\nabla^2 S_* = - \sqrt{n}\, F_{-2}\, S(n-1)\, \psi_n\, e^{imk\xi} \qquad (5.5)$$

$$\nabla^2 \chi_* = - F_1\, \chi(n)\, \psi_n\, e^{imk\xi}$$

$$\eta\nabla^2 \chi_* = - \sqrt{n}\, F_2\, \chi(n)\psi_n\, e^{imk\xi}$$

$$\nabla^2 \Phi_* = - F_1\, \Phi(n)\psi_n\, e^{imk\xi}$$

Noting further that $F_{-2} + \frac{1}{2} = F_o$ and $F_2 - \frac{1}{2} = F_o$ we get finally

$$\frac{d\Phi(n)}{d\tau} = q^2 F_1\, \chi(n)$$

$$\frac{d\chi(n)}{d\tau} = - \Phi(n) + \sqrt{n}\frac{F_o}{F_1}\, S(n-1) + \frac{imk}{F_1}\, \chi(n) \qquad (5.6)$$

$$\frac{dS(n-1)}{d\tau} = - \sqrt{n}\, \frac{F_o}{F_{-1}}\, \chi(n) + \frac{imk}{F_{-1}}\, S(n-1)$$

The system (5.6) is one of the most simple low order systems. We shall show that in spite of its simplicity it has a consistent energy system. For this purpose we define the energy quantities as follows :

available potential energy : $\frac{1}{2}\frac{1}{q^2}\Phi(n)\,\Phi_*(n)$

kinetic energy of divergent motion : $\frac{1}{2} F_1\,\chi(n)\cdot\chi_*(n)$

kinetic energy of non-divergent motion: $\frac{1}{2} F_{-1} S(n-1) S_*(n-1)$

where the asterisk denotes the complex conjugate. Using these definitions we get from (5.6) the following energy equations :

$$\frac{d}{d\tau}\left[\frac{1}{2q^2}\Phi\Phi_*\right] = \tfrac{1}{2}F_1(\chi\Phi_* + \chi_*\Phi)$$

$$\frac{d}{d\tau}\left[\tfrac{1}{2}F_2\,\chi\chi_*\right] = -\tfrac{1}{2}F_1(\chi\Phi_* + \chi_*\Phi) + \tfrac{1}{2}\sqrt{n}\,F_o(S\chi_* + S_*\chi)$$

$$\frac{d}{d\tau}(\tfrac{1}{2}F_{-1}\,S\,S_*) = -\tfrac{1}{2}\sqrt{n}\,F_o(S\chi_* + S_*\chi) \qquad (5.7)$$

which show that $-\frac{1}{2}F_1(\chi\Phi_* + \chi_*\Phi)$ measures the conversion from available potential energy to the kinetic energy of the divergent motion while $-\frac{1}{2}\sqrt{n}\,F_o(S\chi_* + S_*\chi)$ measures the conversion from the latter energy form to the kinetic energy of the non-divergent motion. These energy relations are analogous to those applicable in the case of the general shallow water equations.

Returning to the system (5.6) we derive next the frequency equation by assuming that the time dependent amplitudes are of the form exp $[-ikm\,c\tau]$ when $c$ is the phase speed. Proceeding in the normal way we arrive at the following frequency equation

$$C^3 + \left(\frac{1}{F_1} + \frac{1}{F_{-1}}\right)C^2 - \left(\frac{n}{m^2k^2}\,\frac{Fo^2}{F_1F_{-1}} + \frac{q^2}{m^2k^2}F_1 - \frac{1}{F_1F_{-1}}\right)C - \frac{q^2}{m^2k^2}\,\frac{F_1}{F_{-1}} = 0 \qquad (5.8)$$

which is solved by numerical methods. Figure 14 and Figure 15 show a comparison between the speeds of the gravity waves computed from the exact frequency equation (3.6) and from (5.8) for n = 1 and n = 5. Although there are substantial differences it is nevertheless seen that the highly truncated model gives phase speeds of the correct order of magnitude.

We shall next consider some balanced models. For this purpose we consider the various forms of the balance equation obtained from the second equation in (5.1). A quasi-balanced model will be one where we assume the balance

$$0 = - \nabla^2\Phi_* + \eta\nabla^2 S_* + \frac{\partial S_*}{\partial\eta} = -\nabla^2\Phi_* + \nabla \cdot (\eta\nabla S_*) \qquad (5.9)$$

in which the full beta effect has been included. The corresponding equation in the low order system is

$$0 = - \Phi(n) + \sqrt{n}\,\frac{F_o}{F_1}\,S(n-1) \qquad (5.10)$$

On the other hand, a quasi-geostrophic model will be based on the equation

$$0 = - \nabla^2\Phi_* + \eta\nabla^2 S_* \qquad (5.11)$$

leading to the low order equation

$$0 = - \Phi(n) + \sqrt{n}\,\frac{F_{-2}}{F_1}\,S(n-1) \qquad (5.12)$$

For a quasi-balanced model we get then the following system replacing (5.6) :

$$\frac{d\Phi(n)}{d\tau} = q^2 F_1 \ X(n)$$

$$0 = -\Phi(n) + \sqrt{n}\,\frac{F_o}{F_{-1}}\, S(n-1) \qquad (5.13)$$

$$\frac{dS(n-1)}{d\tau} = -\sqrt{n}\,\frac{F_o}{F_{-1}}\, X(n) + \frac{imk}{F_{-1}}\, S(n-1)$$

which leads to the phase speed

$$C_{q.b.} = -\frac{1}{F_{-1} + \frac{n}{q^2}\frac{F_o^2}{F_1^2}} \qquad (5.14)$$

For a quasi-geostrophic model the equations are :

$$\frac{d\Phi\ (n)}{d\tau} = q^2 F_1 \ X(n)$$

$$0 = -\Phi(n) + \sqrt{n}\,\frac{F_{-2}}{F_1}\, S(n-1) \qquad (5.15)$$

$$\frac{dS(n-1)}{d\tau} = -\sqrt{n}\,\frac{F_2}{F_{-1}}\, X(n) + \frac{imk}{F_{-1}}\, S(n-1)$$

with the phase speed

$$C_{q.g.} = -\frac{1}{F_{-1} + \frac{n}{q^2}\,\frac{F_o^2 - \frac{1}{4}}{F_1^2}} \qquad (5.16)$$

We note that $C_{q.g.}$ and $C_{q.b.}$ are very close to each other as long as $F_o \gg \frac{1}{2}$ . This means that significant differences can occur only on the largest scale, i.e. m and n small. Note also that the formulation (5.15) is energetically inconsistent. The speeds derived in (5.14) and (5.16) may be compared with the speed from a strictly non-divergent model, i.e.

$$\frac{dS}{d\tau} = \frac{ik}{F_{-1}}\, S \qquad (5.17)$$

leading to

$$C_{n.d.} = -\frac{1}{F_{-1}} \tag{5.18}$$

Such a comparison is made in Figures 16, 17 and 18 for the values n = 1, 5 and 10, respectively. As expected $C_{q.b.}$ and $C_{q.g.}$ differ only for n = 1 and for large values of the wave length. Note also that the larger values of n give a considerable reduction of the retrogression for the large wave lengths.

Since we intend to investigate various initialisation procedures we shall need to know the divergence implied by the various models. It is well known that the divergence may be calculated from the equations for a quasi-balanced or quasi-geostrophic model. In the first case we have

$$\frac{d\Phi}{d\tau} = q^2 F_1 \chi$$

$$\frac{dS}{d\tau} = -\sqrt{n}\,\frac{F_o}{F_{-1}}\,\chi + \frac{imk}{F_{-1}}\,S \tag{5.19}$$

$$S = \frac{1}{\sqrt{n}}\,\frac{F_1}{F_o}\,\Phi$$

Upon elimination of the time derivatives we get

$$\chi = ik\,\frac{F_1^2}{q^2 F_1^3\, F_{-1} + nF_o^2\, F_{-1}}\,\Phi \tag{5.20}$$

The corresponding equations for the quasi-geostrophic case are :

$$\frac{d\Phi}{d\tau} = q^2 F_1 \ \chi$$

$$\frac{dS}{d\tau} = - \sqrt{n} \ \frac{F_2}{F_{-1}} \ \chi + \frac{ik}{F_{-1}} \ S \qquad (5.21)$$

$$S = \frac{1}{\sqrt{n}} \ \frac{F_1}{F_{-2}} \ \Phi$$

leading to the following value of :

$$\chi = ik \frac{F_1}{q^2 F_1^2 \ F_{-1} + nF_2 F_{-2}} \ \Phi \qquad (5.22)$$

We shall next investigate the partioning of the initial amplitudes in the stream function, the velocity potential and the geopotential among the various components. From the continuity equation we have the relation

$$\chi = - ik \frac{C}{q^2 F_1} \ \Phi \qquad (5.23)$$

for each value of C.

The vorticity equation leads to

$$S = - \frac{\sqrt{n}}{q^2} \ \frac{F_o C}{F_1 (F_{-1} C + 1)} \ \Phi \qquad (5.24)$$

for each value of C. (5.24) has been obtained using not only the continuity equation, but also (5.23) . The two expressions (5.23) and (5.24) lead to a total of six relations because C has three values. In addition we have the three relations expressing that the sum of the amplitudes of S, $\chi$ and $\Phi$ must be equal to the initial amplitude, i.e.

$$\begin{aligned} S_1 + S_2 + S_3 &= S_o \\ \chi_1 + \chi_2 + \chi_3 &= \chi_o \\ \Phi_1 + \Phi_2 + \Phi_3 &= \Phi_o \end{aligned} \tag{5.25}$$

When (5.23) and (5.24) are substituted in (5.25) we get three linear inhomogeneous equations in $\Phi_1, \Phi_2$ and $\Phi_3$. The determinant of these equations are :

$$\Delta = ik \frac{n}{q^4} \frac{F_o}{F_1^2} \left\{ \frac{C_1(C_2 - C_3)}{F_{-1}\ C_1+1} + \frac{C_2(C_3 - C_1)}{F_{-1}C_2+1} + \frac{C_3(C_1 - C_2)}{F_{-1}\ C_3+1} \right\}$$

and the solution for $\Phi_1$ is

$$\Phi_1 = \frac{1}{q^2 F_1 \Delta} \left\{ -\ ik\ (C_2 - C_3)\ S_o + \sqrt{n}\ F_o \left( \frac{C_2}{F_{-1}\ C_2+1} - \frac{C_3}{F_{-1}\ C_3+1} \right) \chi_o \right.$$

$$\left. +\ ik \frac{\sqrt{n}}{q^2} \frac{F_o}{F_1}\ C_2 C_3 \left( \frac{1}{F_{-1}\ C_2+1} - \frac{1}{F_{-1}\ C_3+1} \right) \Phi_o \right\} \tag{5.26}$$

The solutions for $\Phi_2$ and $\Phi_3$ are obtained by cyclic permutations among $C_1$, $C_2$ and $C_3$. The values of $\chi_1$ $\chi_2$ and $\chi_3$ are obtained from (5.23) and those of $S_1$, $S_2$ and $S_3$ from (5.24).

We shall investigate four different cases. The first case will be called the geostrophic case because we shall assume that

$$S_o = \frac{1}{\sqrt{n}} \frac{F_1}{F_{-2}} \Phi_o , \quad \chi_o = 0 \tag{5.27}$$

The next case is called the balanced case with the initial condition

$$S_o = \frac{1}{\sqrt{n}} \frac{F_1}{F_o} \Phi_o \quad , \chi_o = 0 \tag{5.28}$$

The third case is a generalisation of the first because the relation between $S_o$ and $\Phi_o$ is as expressed in (5.27), but

$$\chi_o = ik \frac{1}{q^2 F_1 F_{-1} + n \frac{F_2 F_{-2}}{F_1}} \Phi_o \tag{5.29}$$

The fourth case is a similar generalisaiton of the second case. The relation between $S_o$ and $\Phi_o$ is given in (5.28), but

$$\chi_o = ik \frac{1}{q^2 F_1 F_{-1} + n \frac{F_o^2 F_{-1}}{F_1^2}} \Phi_o \tag{5.30}$$

## 6. Numerical Results

The four cases mentioned in Section 5 have been investigated. In this section we shall compare the results obtained from the balanced relation without initial divergence (5.28) and the balanced relation with the initial divergence completed from (5.30). The results from the quasi-geostrophic relation (5.27) without initial divergence are very similar to those obtained from the balanced case with no divergence. Similarly, the results based on (5.29) are quite close to those based on (5.30).

Fig. 19a and b compare $\Phi_1/\Phi_o$ for the two cases. While the ratio remains quite small for the case of no divergence, Fig. 19a, obtaining a maximum value of 0.06 it is even smaller in Fig. 19b where the largest value is 0.02.

The initial divergence has thus reduced the amplitude in the gravity mode by a large amount. The improvement is even larger in the other gravity mode as seen from Fig. 20 a and b. The ratio $\Phi_3/\Phi_o$ for the Rossby mode is less than unity for both cases, but while the ratio for the case of no divergence goes down to about 0.9 it is never less than 0.96 when the initial divergence is added, as illustrated in Fig. 21a and b. The distributions of $S_1/S_o$, $S_2/S_o$ and $S_3/S_o$ for the two cases are quite similar to those for the geopotential.

For these cases we also computed the ratios $\chi_1/S_1$, $\chi_2/S_2$ and $\chi_3/S_3$. As expected we find for the two gravity modes, i.e. $\chi_1/S_1$ and $\chi_2/S_2$, that the ratio is everywhere larger than unity and becomes very large when the scale is small. The ratio $\chi_3/S_3$ corresponding to the Rossbv mode is smaller than unity as illustrated in Fig. 22, but attains values as large as 0.3 on the largest scale. It is of interest to note that the ratio $\chi_i/S_i$, i = 1,2,3 is independent of the initial conditions as seen from (5.23) and (5.24) from which we obtain

$$\frac{\chi_i}{S_i} = ik \frac{F_{-1} C_i + 1}{\sqrt{n}\, F_o} \qquad (6.1)$$

## 7. Concluding Remarks

The main purpose of the paper is to report on the partitioning of the initial amplitudes among gravity waves and Rossby waves for various initial conditions for a model based on the shallow water equations.

The geometry used is that of a plane on which the Coriolis parameter varies linearly with the meridional coordinate. Since this variation is kept in all terms the geometry is different from the ordinary beta-plane. It turns out that the eigenfunctions are of the type $\psi_n \exp(ikmx)$ where $\psi_n$ is a product of a Hermite polynomial and an exponential function. The wave speed is thus a function of both the zonal and the meridional scale.

Section 3 considers the linear problem, and Section 4 contains some numerical examples showing that an initial condition of no divergence in the horizontal windfield results in rather small amplitudes in the gravity waves except at very large scales.

Section 5 considers the linear problem based on the vorticity, divergence and continuity equations reduced to a low order system. Four initial conditions are used :

- (i) geostrophic vorticity, no divergence
- (ii) vorticity from the balance equation, no divergence
- (iii) geostrophic vorticity; divergence from a quasi-geostrophic model
- (iv) vorticity from the balance equation; divergence from a quasi-balanced model.

The results from (i) and (ii) are very similar as are those from (iii) and (iv). Section 6 contains a comparison of the initial conditions (ii) and (iv).

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## Appendix 1.

The solutions to the basic equation (3.4) are of the type

$$\psi_n(z) = e^{-\frac{1}{4}z^2}\frac{He_n(z)}{\sqrt{n!}\sqrt{2\pi}} \tag{A1.1}$$

where $\psi_n(z)$ has been normalised in such a way that

$$\int_{-\infty}^{\infty} \psi_n(z)^2 \, dz = 1 \tag{A1.2}$$

The functions $\psi_n(z)$ are orthogonal in the sense that

$$\int_{-\infty}^{\infty} \psi_m(z)\,\psi_n(z)\,dz = \begin{cases} 0 \;, & m \neq n \\ 1 \;, & m = n \end{cases} \tag{A1.3}$$

The function $He_n(z)$ is a Hermite polynomial related to the basic Hermite polynomial by the formula

$$He_n(z) = 2^{-\frac{1}{2}n}\, H_n\left(\frac{z}{\sqrt{2}}\right) \tag{A1.4}$$

For the basic polynomial we have the formula

$$H_{n+1}(z) = 2z\, H_n(z) - 2n\, H_{n-1}(z) \tag{A1.5}$$

Using (A1.4) in (A1.5) we may easily derive the relation

$$He_{n+1}(z) = z\, He_n(z) - n\, He_{n-1}(z) \tag{A1.6}.$$

Furthermore, substituting from (A1.1) into (A1.6) we can derive the relation

$$\sqrt{n+1}\;\psi_{n+1}(z) = z\psi_n(z) - \sqrt{n}\;\psi_{n-1}(z) \tag{A1.7}$$

which is used repeatedly in modifying the Coriolis terms in the equations of motion.

We shall also need a formula for the differential of $\psi_n(z)$. Taking our starting point in

$$\frac{dH_n(z)}{dz} = 2n\ H_{n-1}(z) \qquad \text{(A1.8)}$$

we may first use (A1.4) to derive that

$$\frac{dHe_n(z)}{dz} = n\ He_{n-1}(z) \qquad \text{(A1.9)}$$

Into this formula we introduce the definition of $\psi_n(z)$ from (A1.1). We obtain then

$$\frac{d\psi_n}{dz} = \sqrt{n}\ \psi_{n-1}(z) - \tfrac{1}{2}z\ \psi_n(z) \qquad \text{(A1.10).}$$

(A1.10) may be expressed in a simpler form by using (A1.7) for $z\ \psi_n(z)$. We get finally :

$$\frac{d\psi_n}{dz} = \tfrac{1}{2}\ \sqrt{n}\ \psi_{n-1}(z) - \tfrac{1}{2}\ \sqrt{n+1}\ \psi_{n+1}(z) \qquad \text{(A1.11)}$$

The most important formulas in this section are (A1.7) and A1.11).

## Appendix 2.

As seen in the text it is a necessity to calculate certain integrals depending on triple products of the functions $\psi_n(z)$. We must first be able to calculate the Hermite polynomials. Our starting point is the fact that

$$He_o(z) = 1; \quad He_1(z) = z \tag{A2.1}$$

Additional values of $He_n(z)$ may then be calculated from (A1.6) or (A1.7). We find that

$$He_2(z) = z^2-1, \quad \psi_2(z) = (2\pi)^{-\frac{1}{4}}\, 2^{-\frac{1}{2}}\, (z^2-1)e^{-\frac{1}{4}z^2}$$

$$He_3(z) = z^3- 3z, \quad \psi_3(z) = (2\pi)^{-\frac{1}{4}}\, 6^{-\frac{1}{2}}\, (z^3-3z)e^{-\frac{1}{4}z^2}$$

$$He_4(z) = z^4-6z^2+3, \quad \psi_4(z)= (2\pi)^{-\frac{1}{4}}\, 24^{-\frac{1}{2}}(z^4-6z^2 +3)e^{-\frac{1}{4}z^2}$$

(A2.2)

$$He_5(z) = z^5- 10z^3+15z, \psi_5(z) = (2\pi)^{-\frac{1}{4}}120^{-\frac{1}{2}}(z^5-10z^3+15z)e^{-\frac{1}{4}z^2}$$

$$He_6(z) =z^6-15z^4 +45z^2-15, \psi_6(z)= (2\pi)^{-\frac{1}{4}}\, 720^{-\frac{1}{2}}(z -15z - 15)e^{-\frac{1}{4}z^2}$$

The interaction integral is defined as follows:

$$I(n_1,n_2,n) = \int_{-\infty}^{\infty} \psi_{n1}(z)\, \psi_{n2}(z)\, \psi_n(z)dz \tag{A2.3}$$

Since $\psi_n(z)$ is symmetric around $z = 0$ when n is even and antisymmetric when n is odd it follows that I is zero whenever $n_1+n_2+n$ is odd. A non-zero value of the interaction integral will consist of integrals of the form

$$\int_{-\infty}^{\infty} z^{2n}\, e^{-az^2}\, dz = 2\int_0^{\infty} z^{2n}\, e^{-az^2}dz = 2\, \frac{1.3\ldots(2n-1)}{2^{n+1}\, a^n}\sqrt{\frac{\pi}{a}}$$

(A2.4)

It is thus straightforward to calculate the interaction integrals. We shall calculate a few which are used in a low-order model. They are listed below :

$I\ (0,0,0) = 0.5157$

$I\ (2,2,2) = 0.2634$

$I\ (2,3,3) = 0.2206$

$I\ (3,3,4) = 0.1910$

$I\ (2,2,4) = 0.0799$

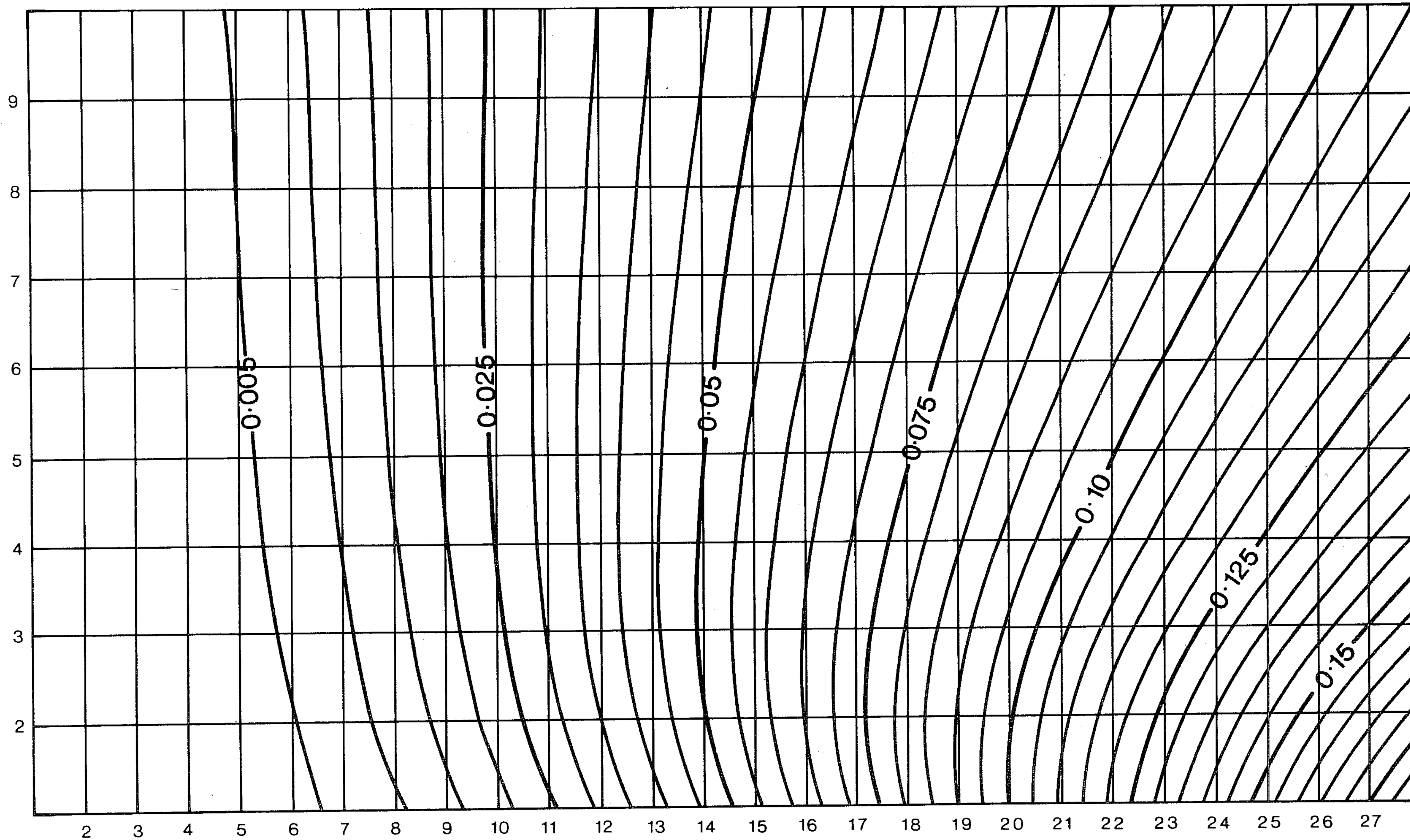


Fig.1 The ratio $V_1/V_o$ as a function of zonal wavelength in $10^6$m and meridional index n. No initial divergence.

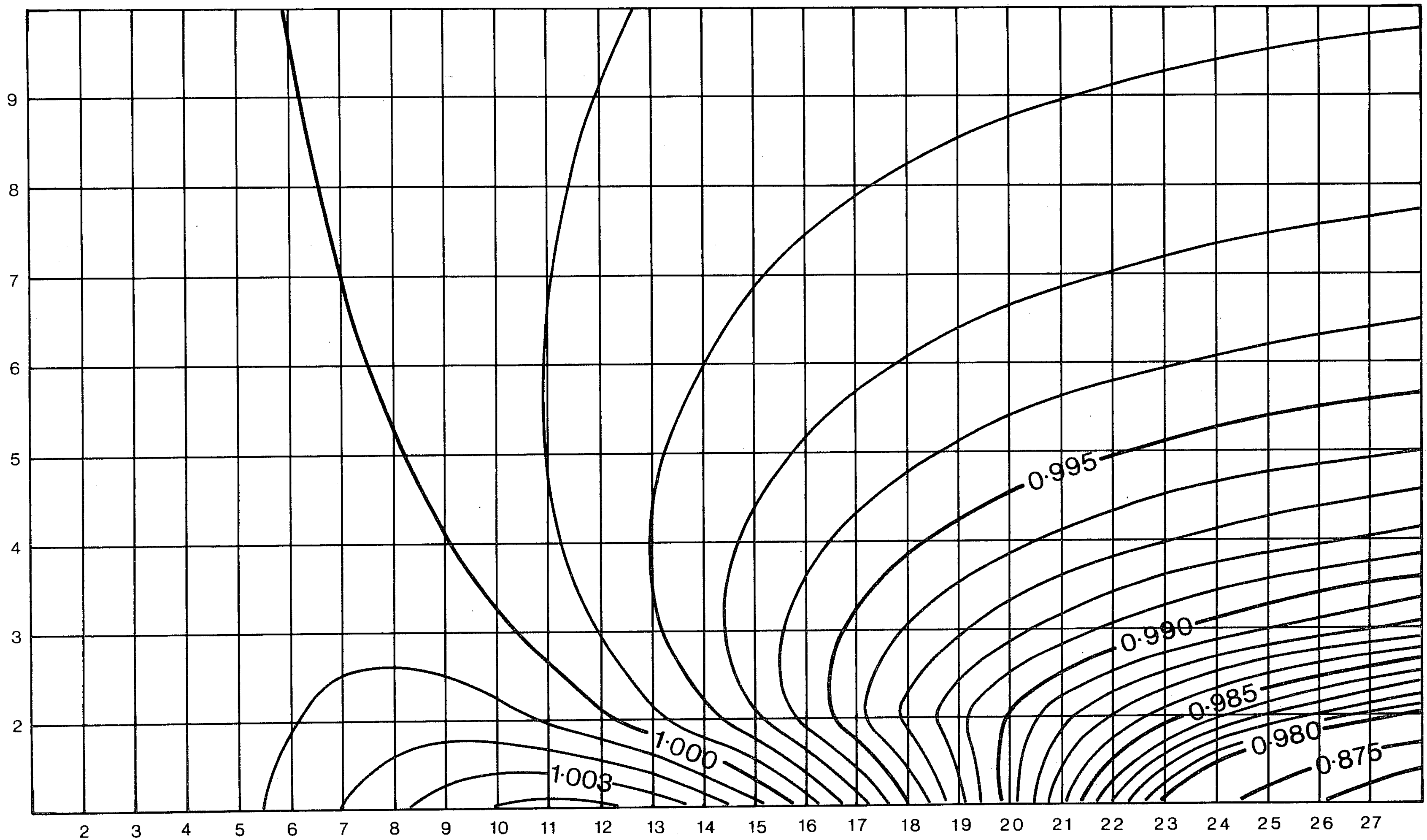


Fig. 2 The ratio $V_3/V_o$, no initial divergence.

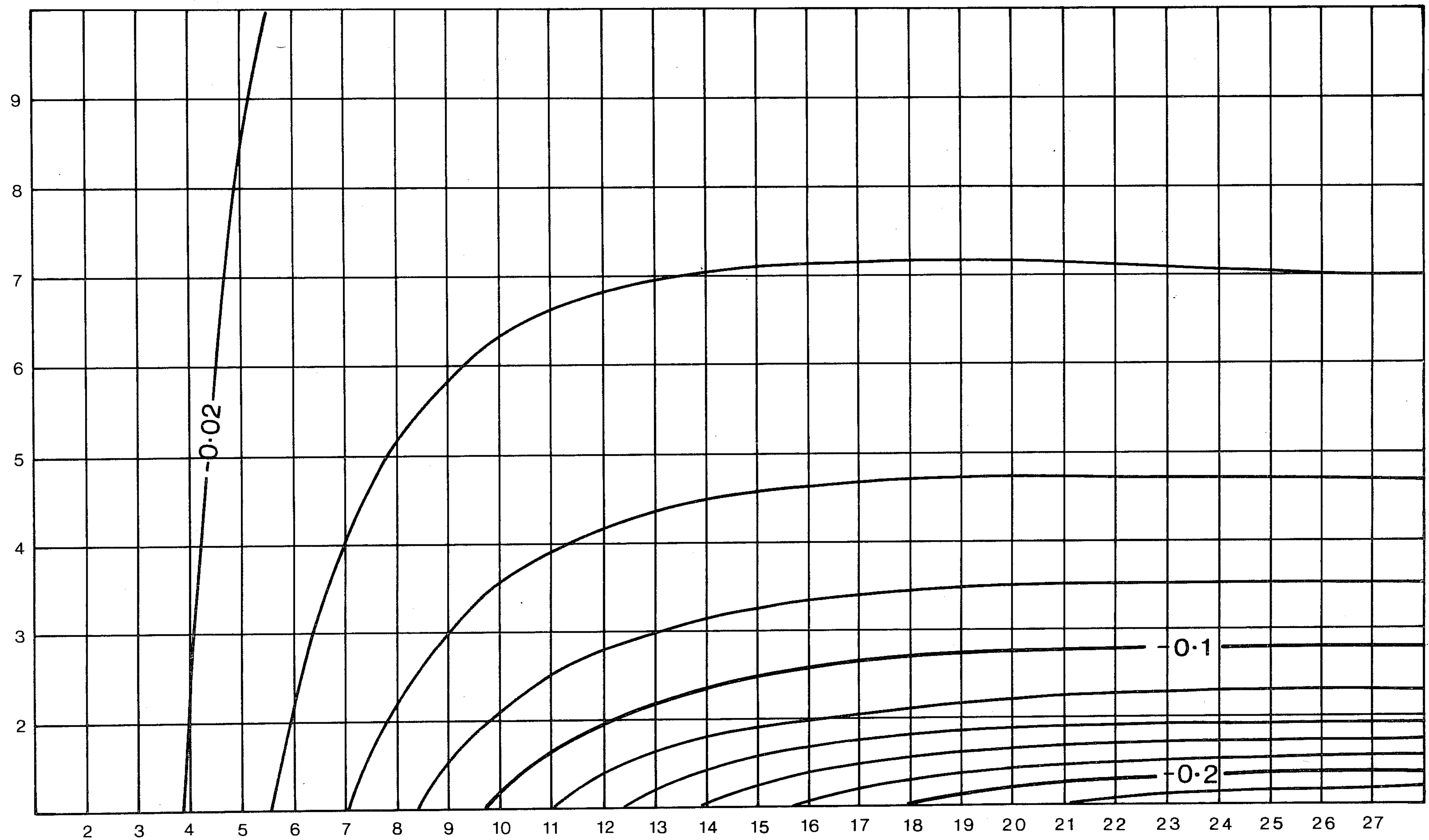


Fig. 3 The ratio $U_{12}/U_{10}$, no initial divergence.

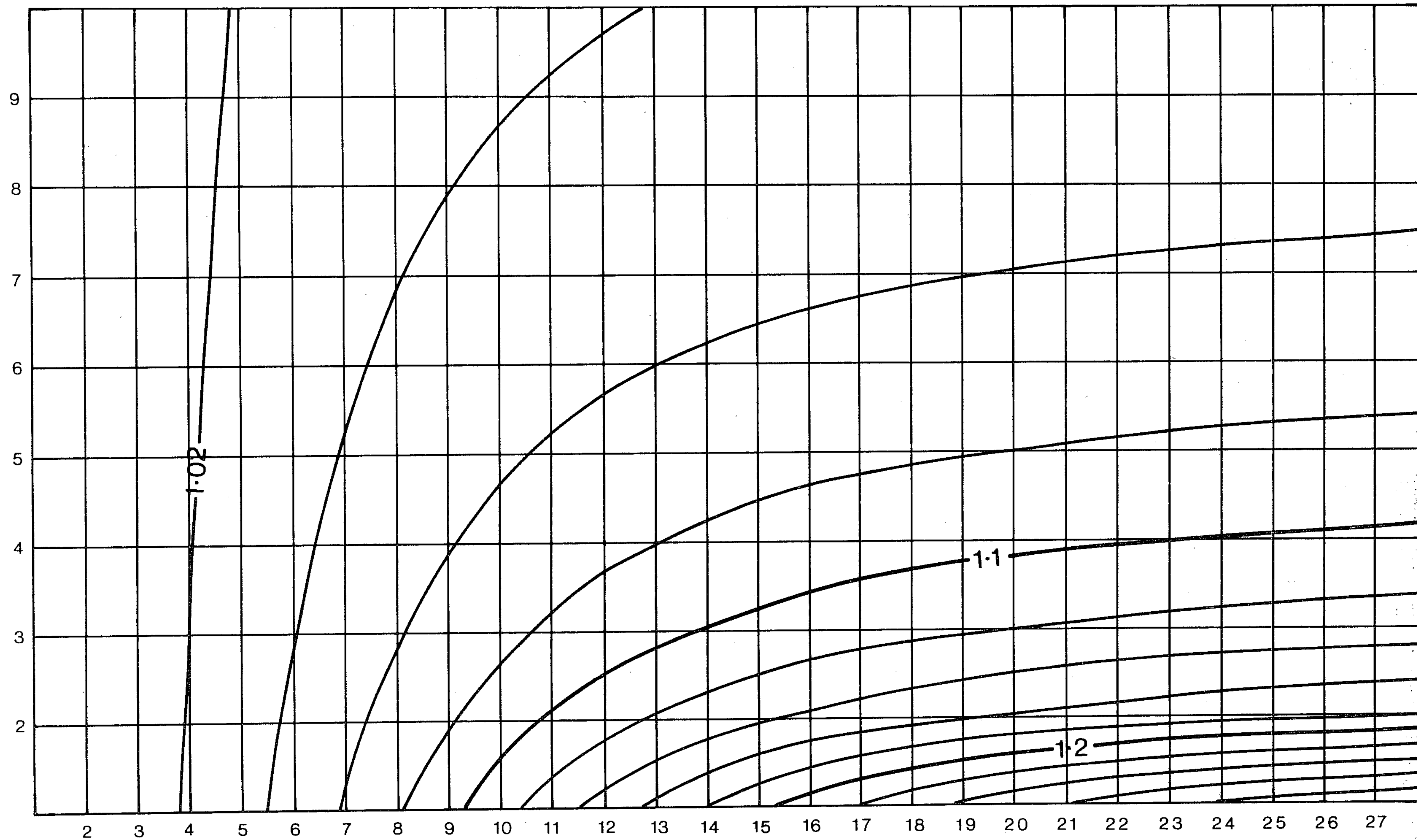


Fig. 4 The ratio $U_{13}/U_{10}$, no initial divergence.

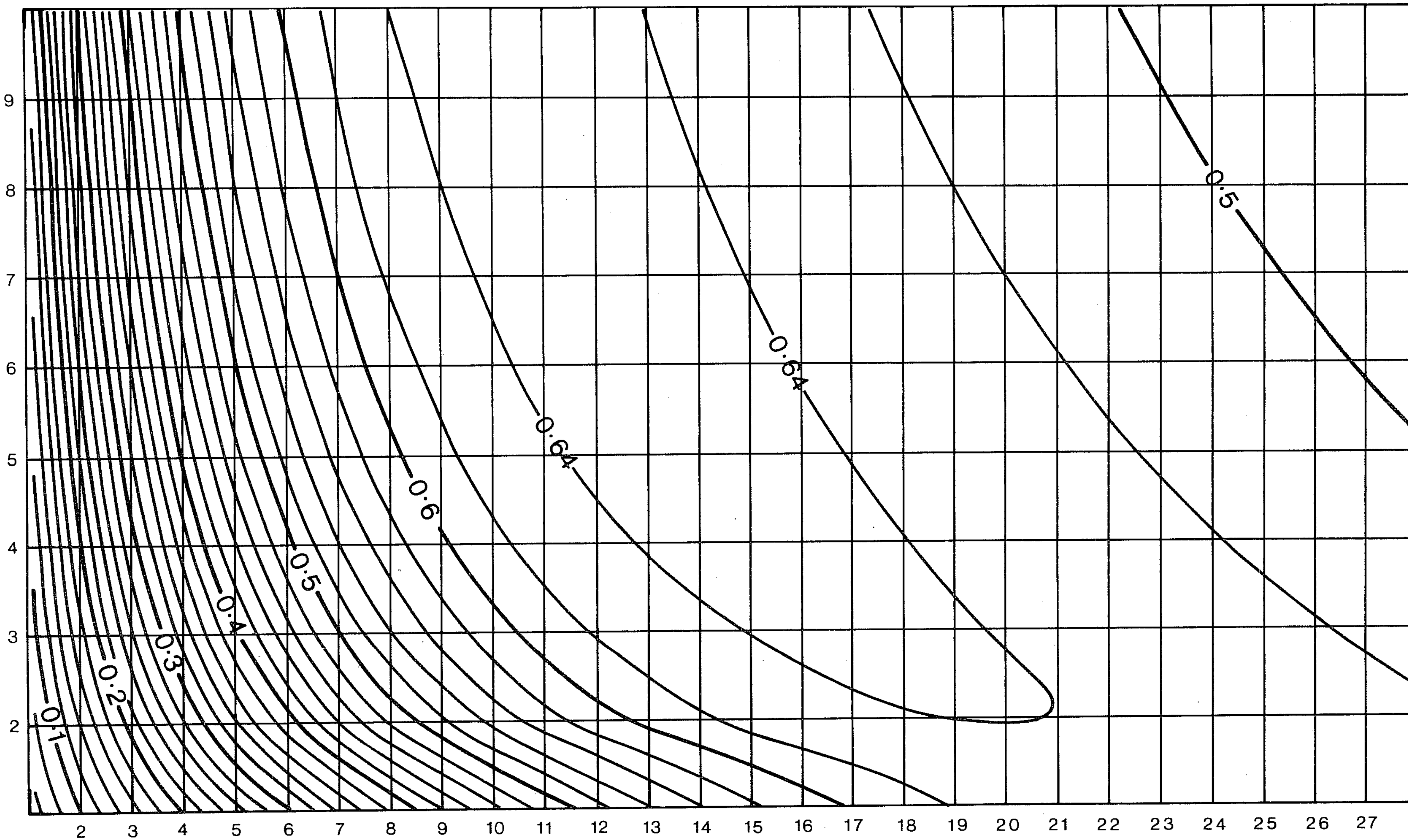


Fig. 5 The ratio $V_2/U_{10}$; initial condition : $U_{10} \neq 0$, $U_{20} = V_o = 0$

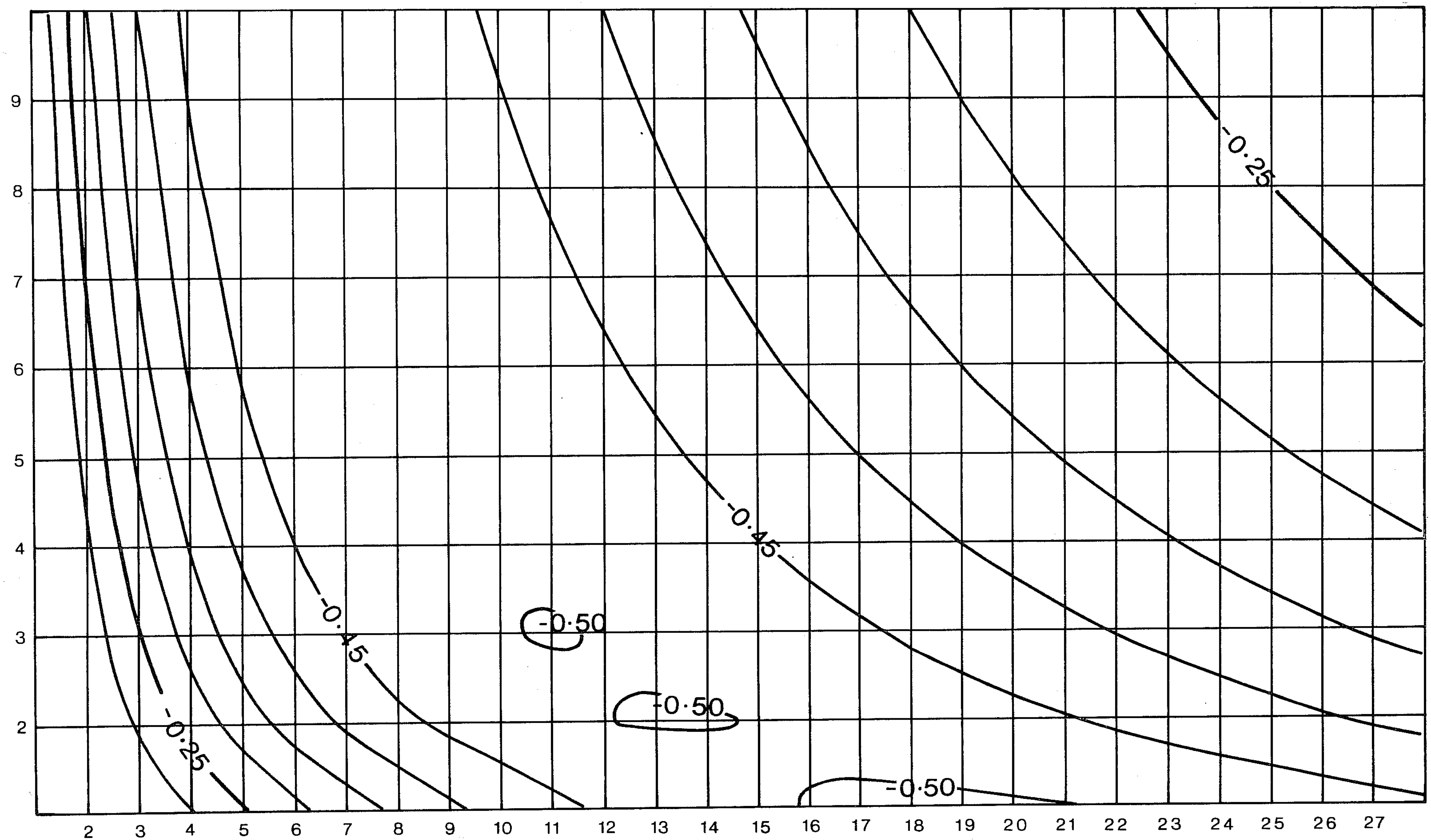


Fig. 6 The ratio $V_3/U_{10}$, initial condition : $U_{10} \neq 0$, $U_{20} = V_o = 0$

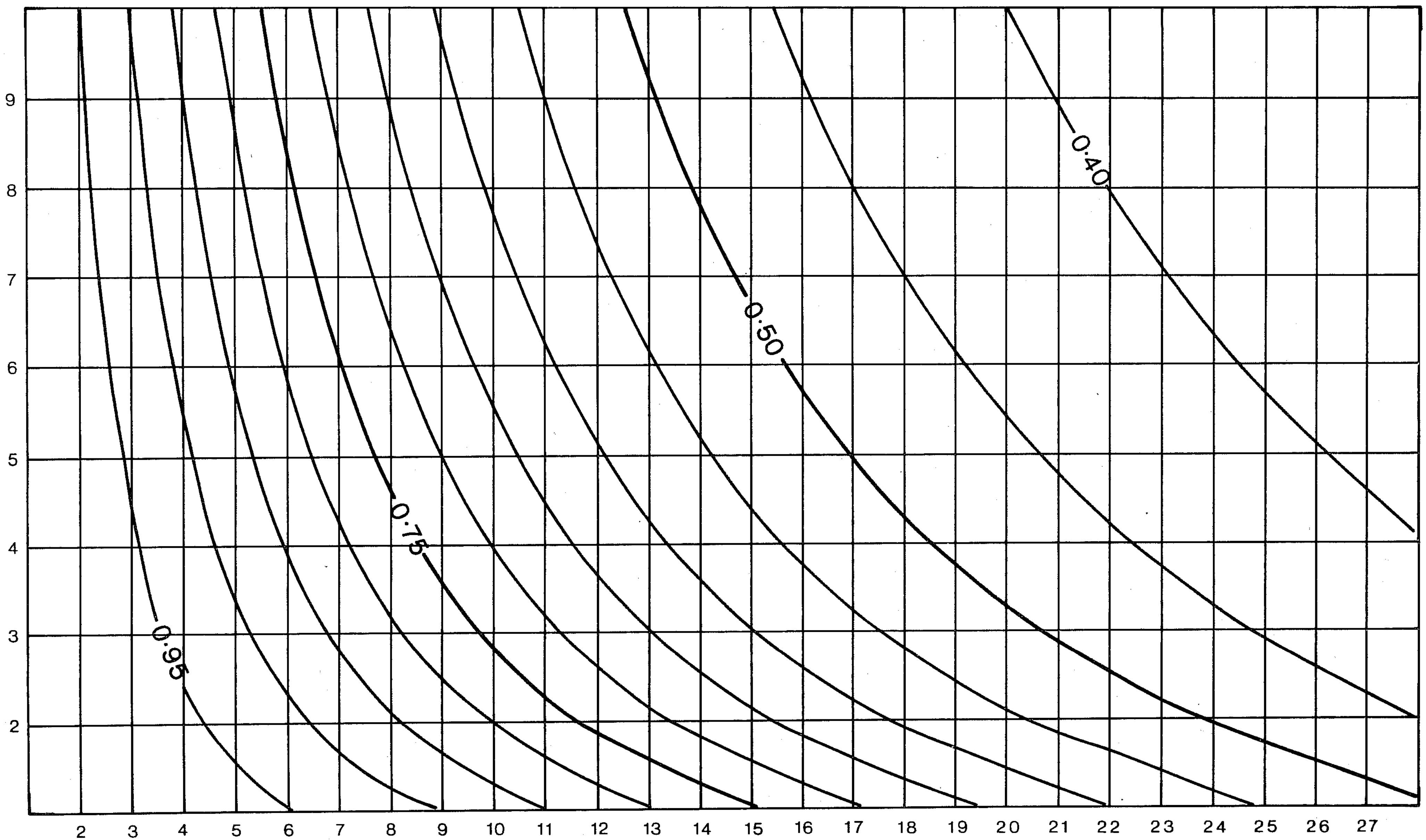


Fig. 7 The ratio $U_{12}/U_{10}$, initial condition : $U_{10} \neq 0$, $U_{20} = V_o = 0$

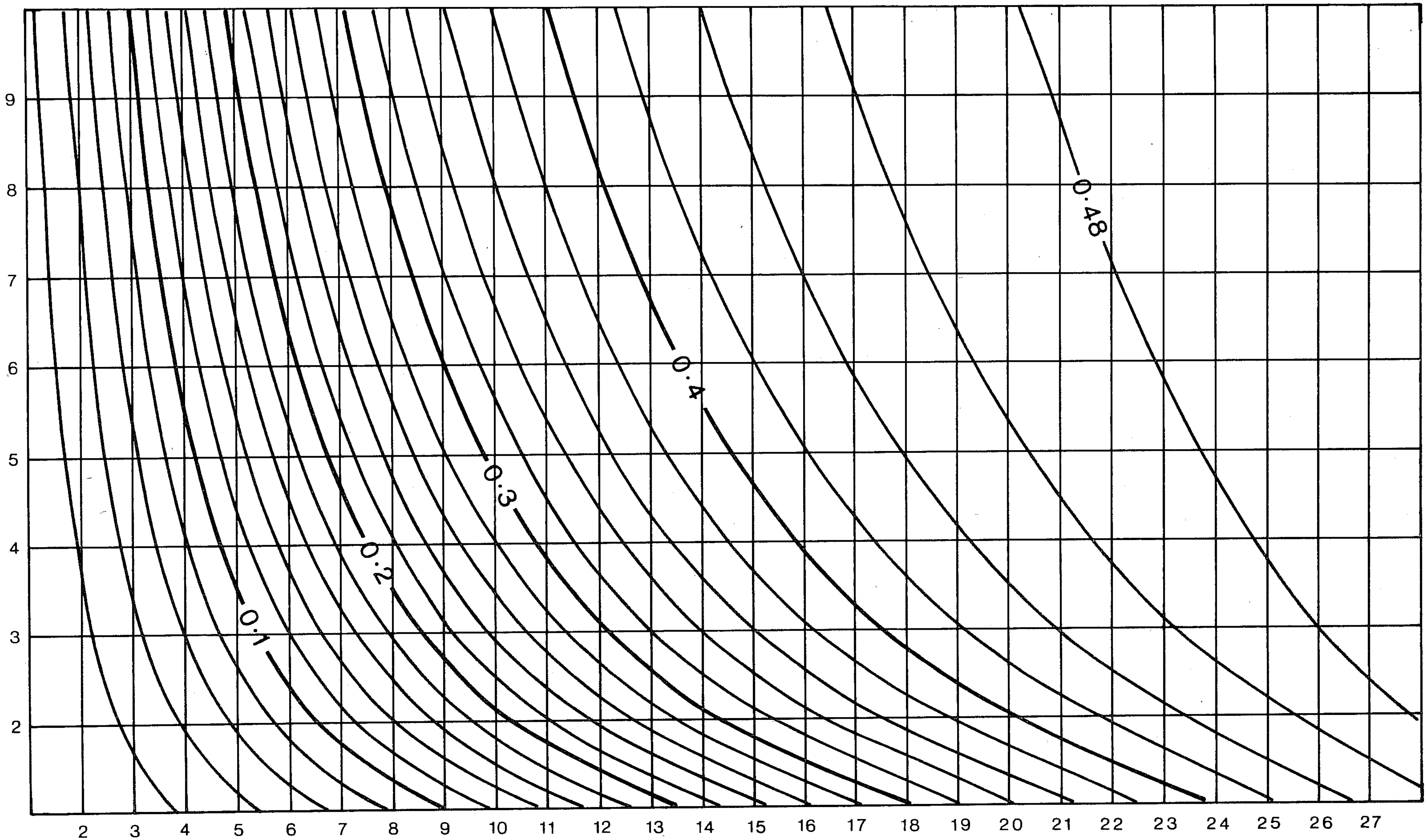


Fig. 8 The ratio $U_{13}/U_{10}$, initial condition : $U_{10} \neq 0$, $U_{20} = V_o = 0$

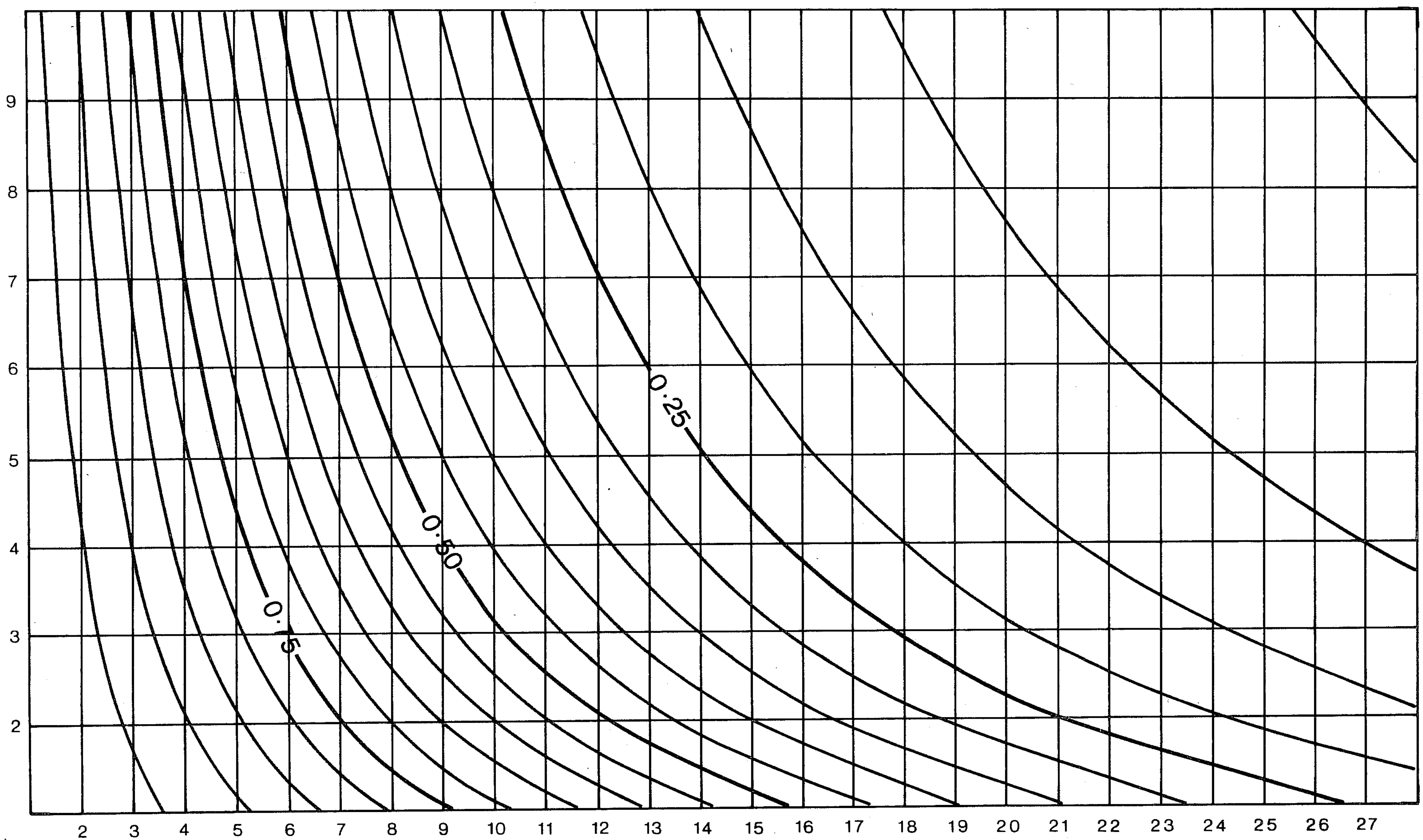


Fig.9 The ratio $V_3/V_o$, initial condition : $V_o \neq 0$, $U_{10} = U_{20} = 0$

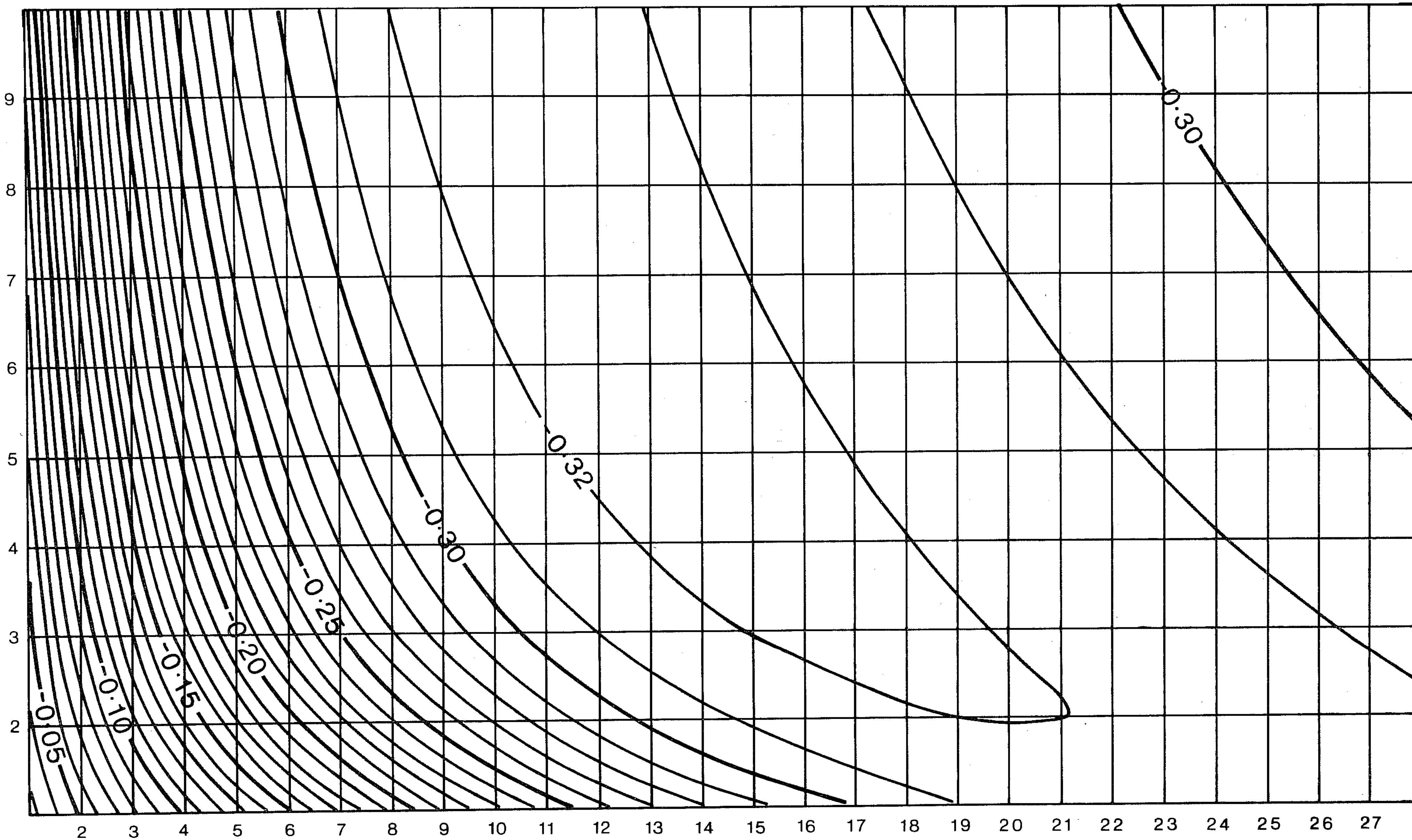


Fig.10 The ratio $U_{12}/V_o$, initial condition : $V_o \neq 0$, $U_{10} = U_{20} = 0$

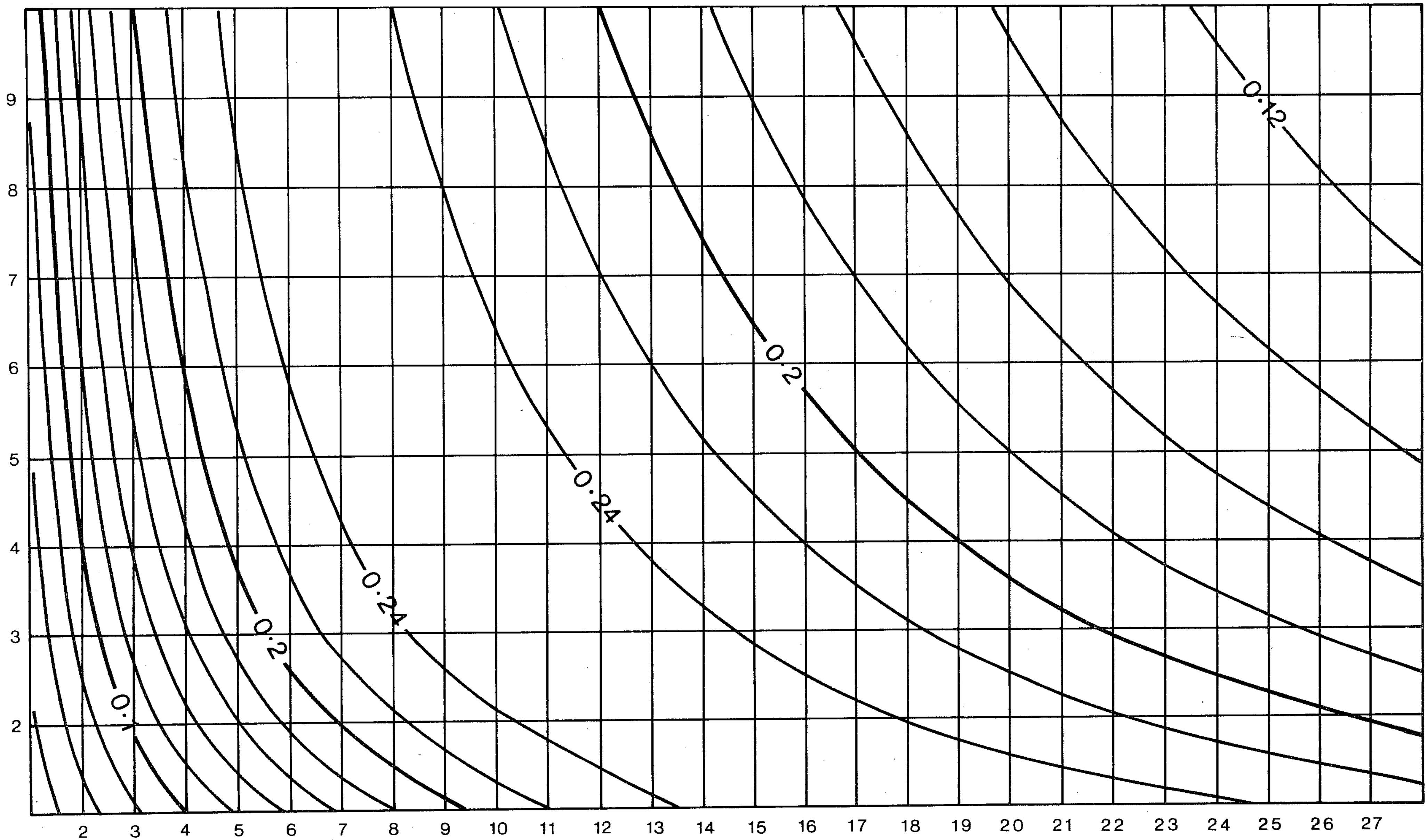


Fig.11 The ratio $U_{13}/V_o$, initial condition : $V_o \neq 0$, $U_{10} = U_{20} = 0$

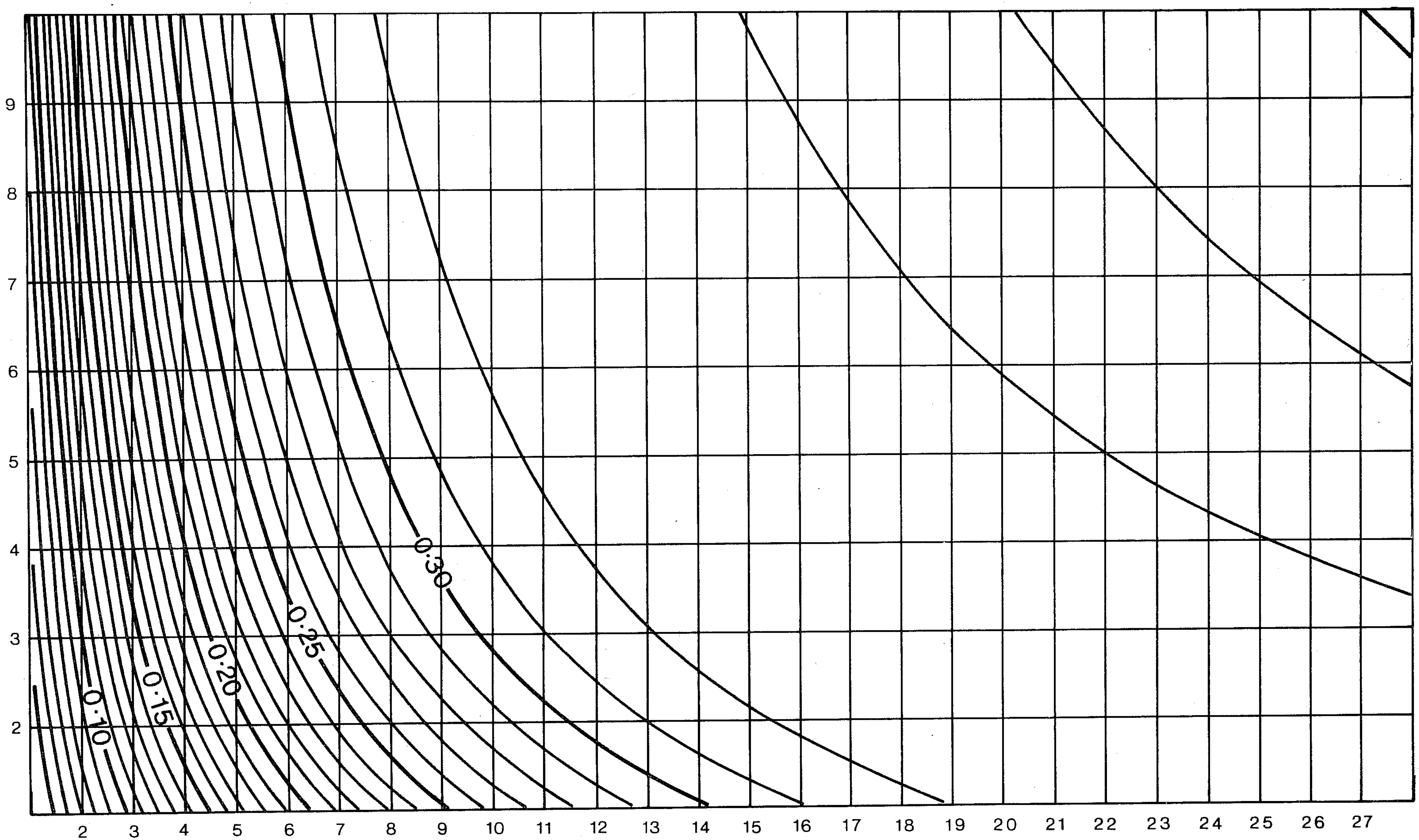


Fig.12 The ratio $U_{21}/V_o$, initial condition : $V_o \neq 0$, $U_{10} = U_{20} = 0$

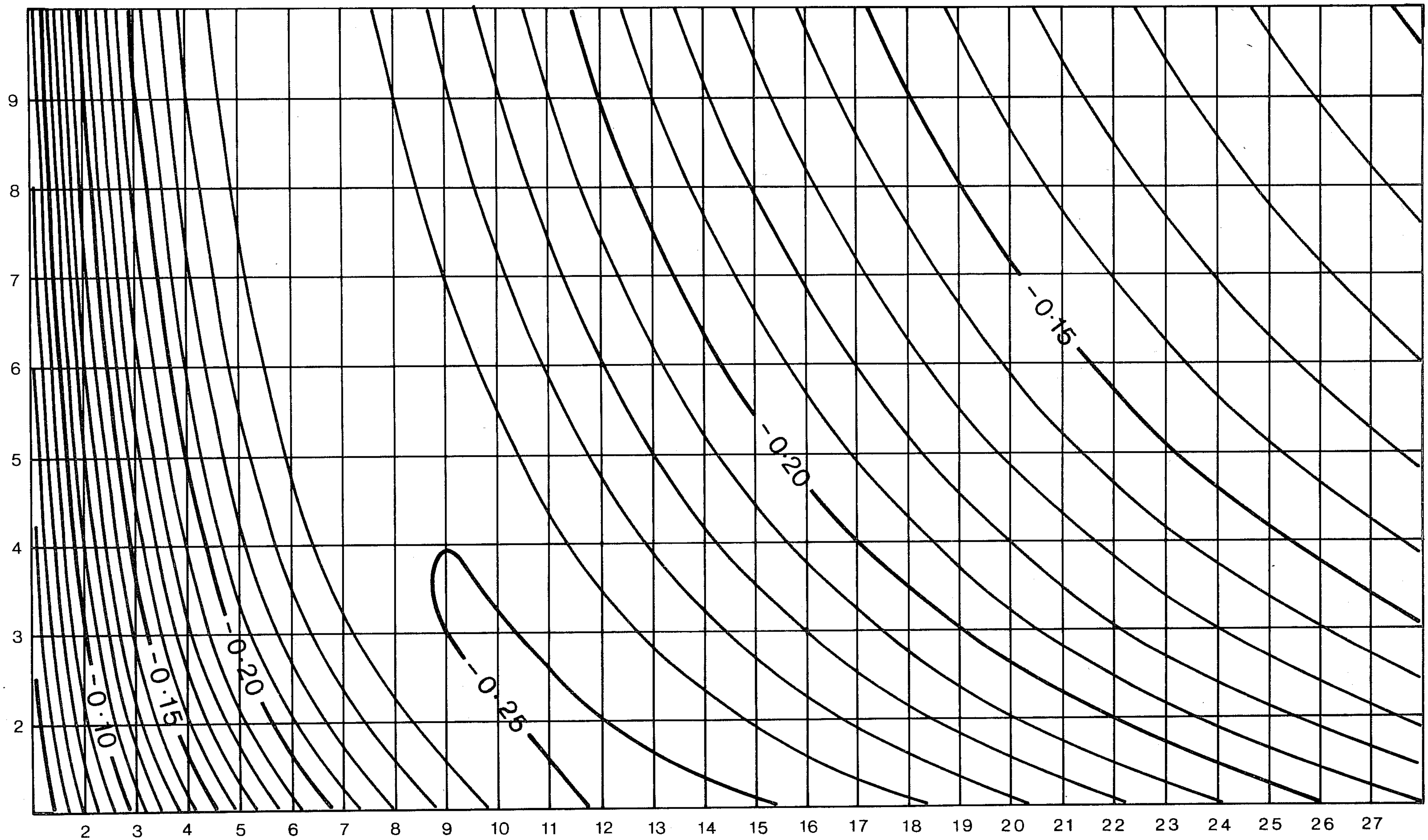


Fig.13 The ratio $U_{23}/V_o$, initial condition : $V_o \neq 0$, $U_{10} = U_{20} = 0$

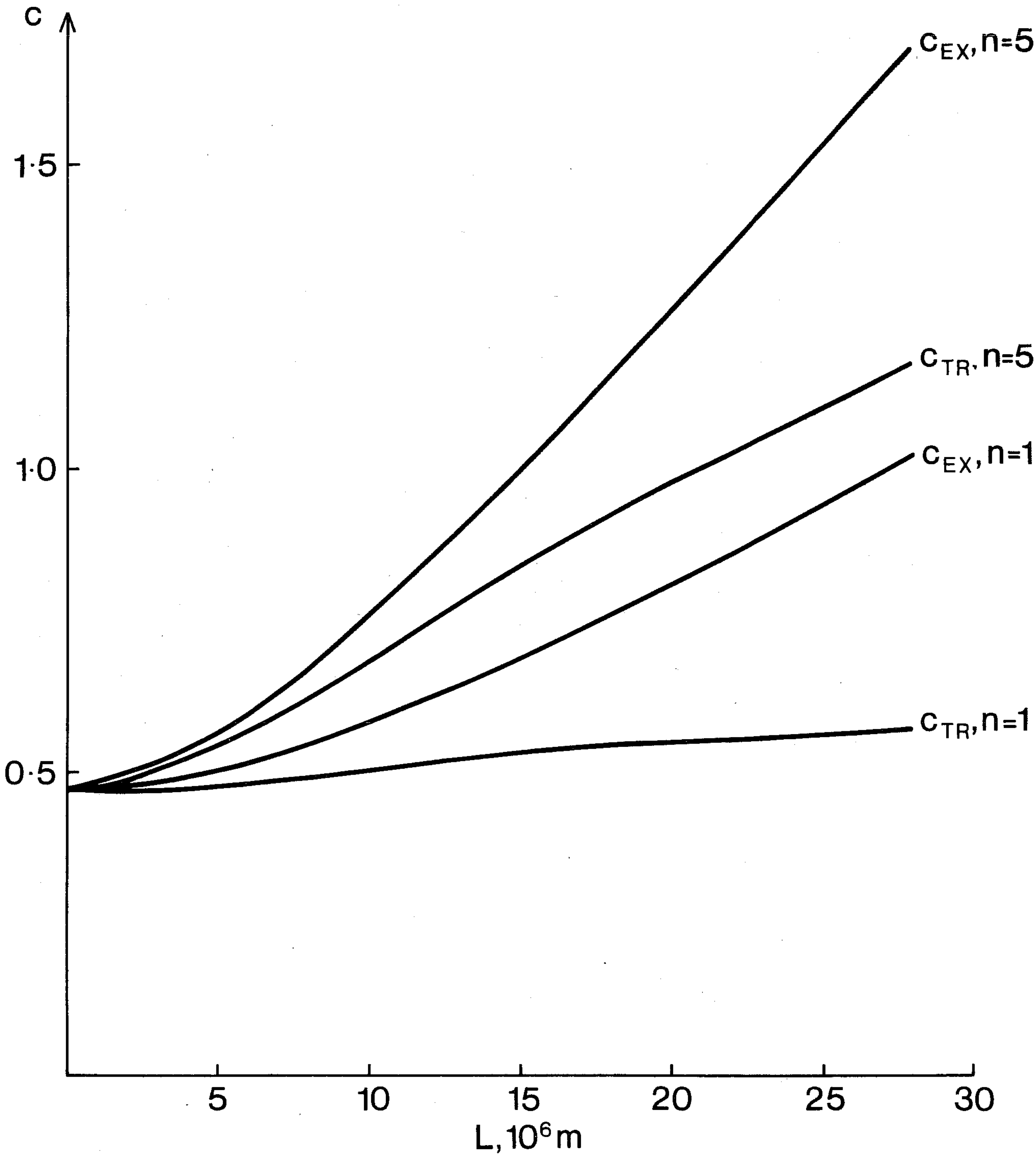


Fig.14 Comparison between wave speeds for n = 1 and n = 5 from the exact frequency equation and the frequency equation for the low order system for gravity waves with a positive speed. The ordinate is the non-dimensional wave speed and the abscissae the wave length.

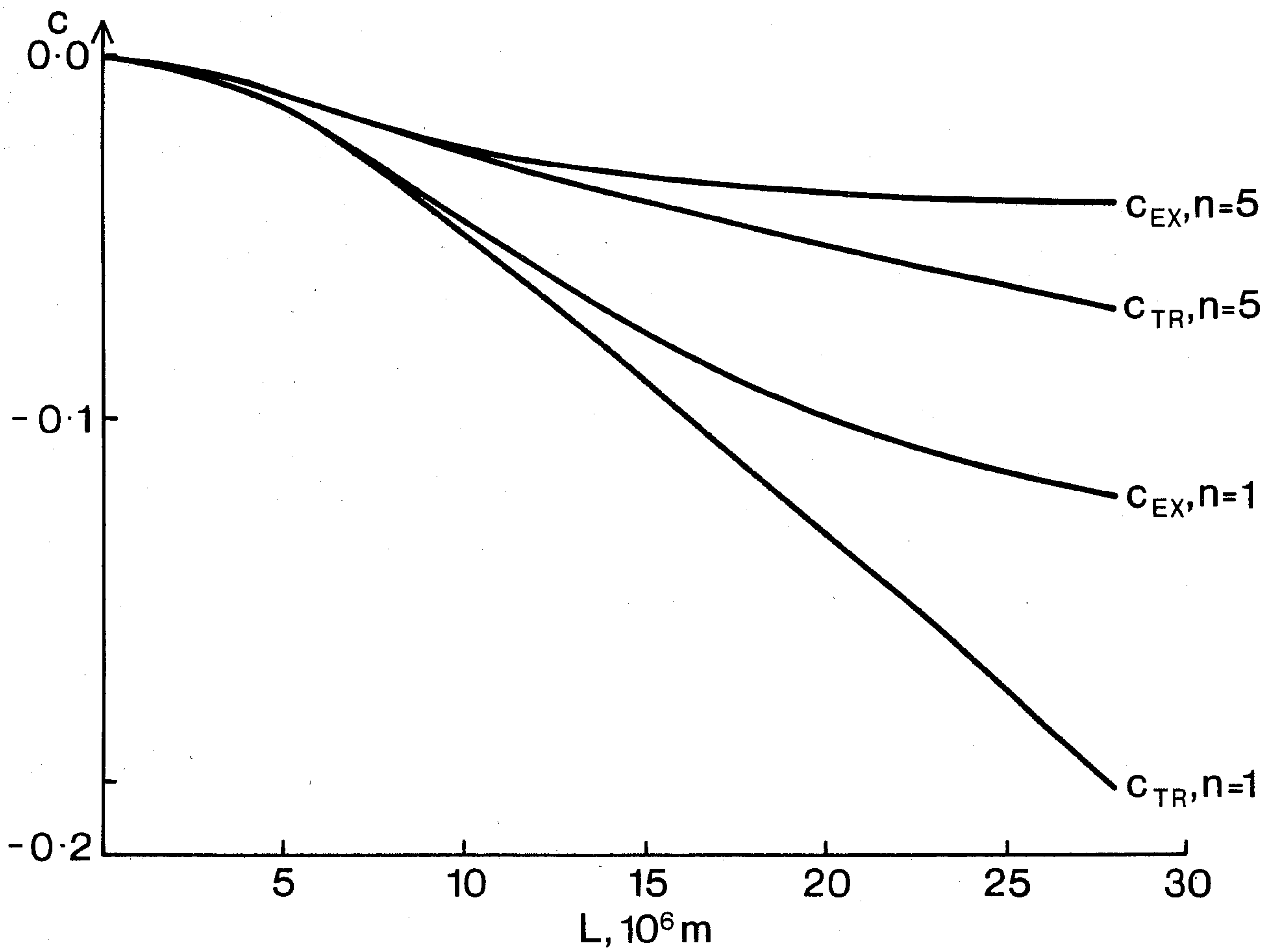


Fig.15 Comparison between wave speeds for n = 1 and n = 5 from the exact frequency equation and the frequency equation for the low order system for Rossby waves.

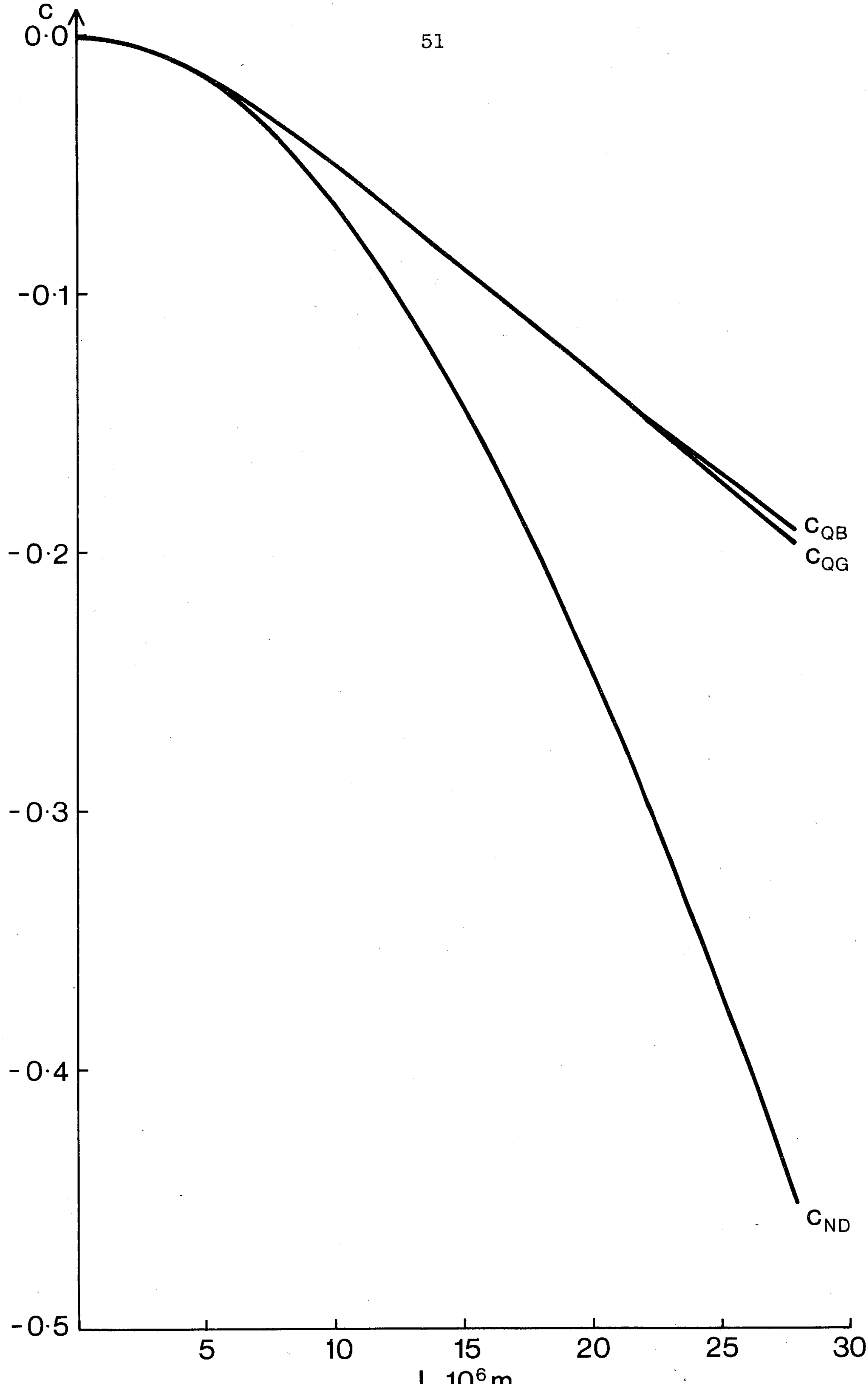


Fig.16 Comparison between wave speeds for n = 1 for a non-divergent (N.D.), a quasi-geostrophic (Q.G.) and a quasi-balanced (Q.B.) model.

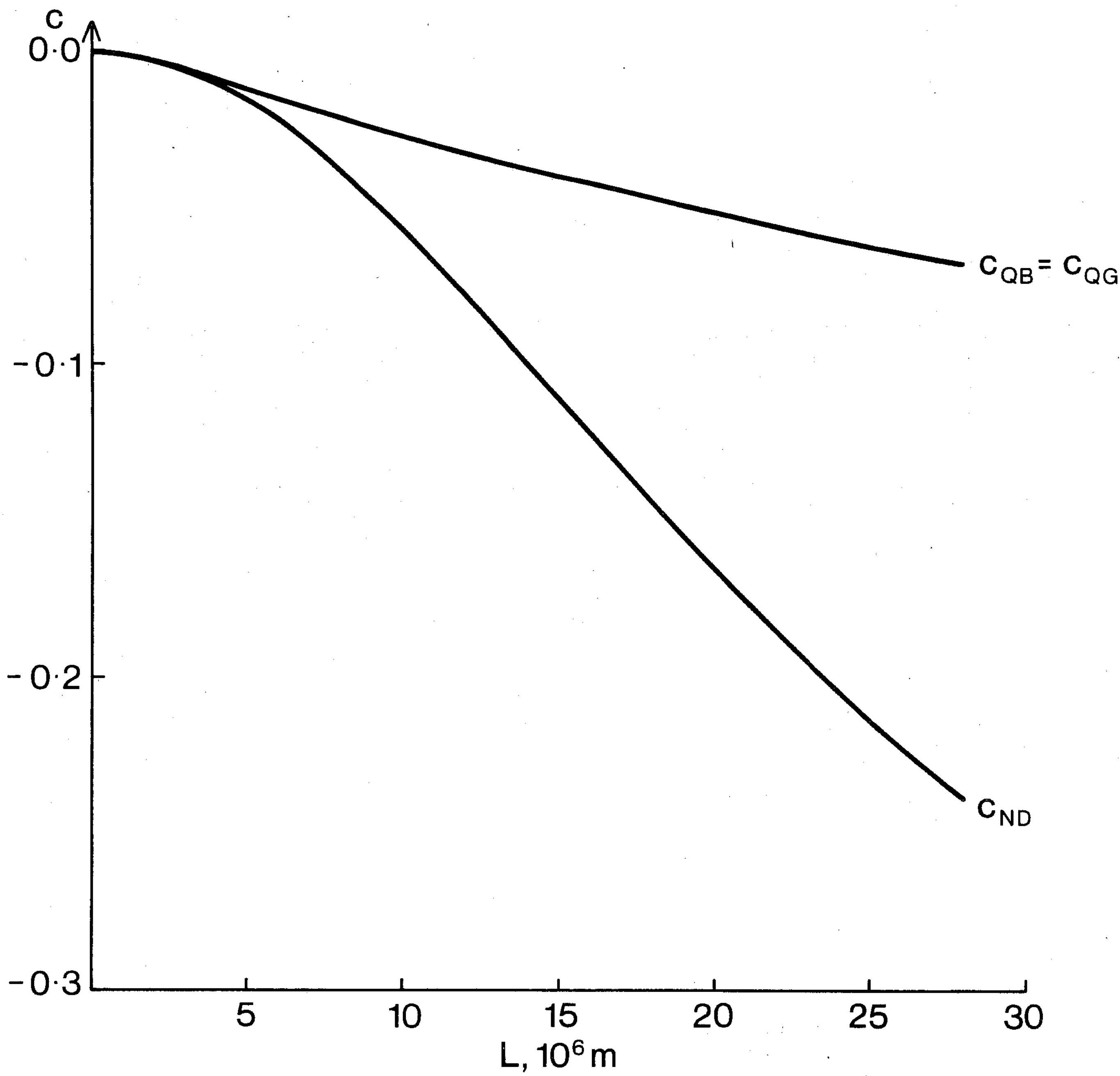


Fig. 17  n = 5, otherwise as Fig. 16

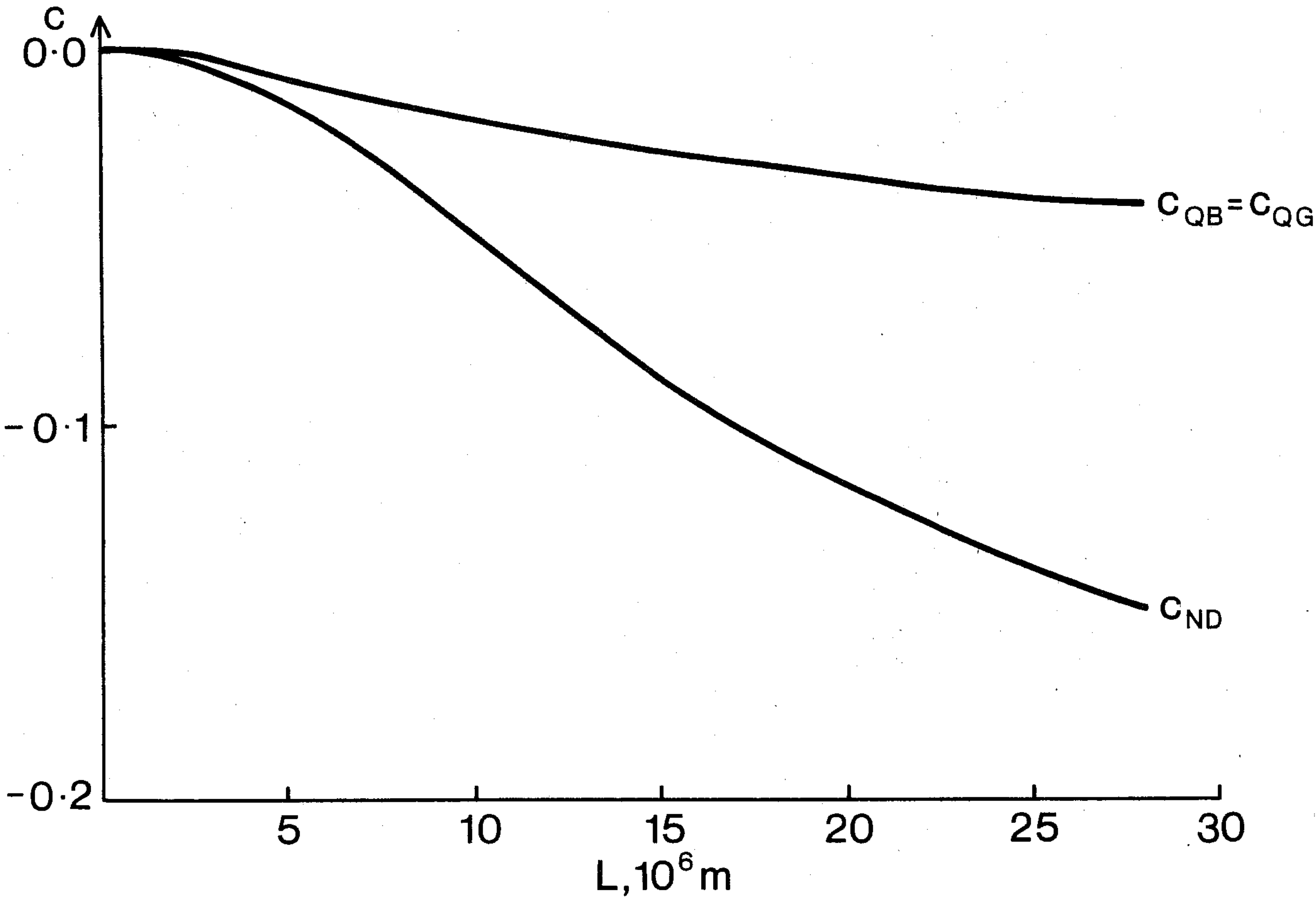


Fig. 18   n = 10, otherwise as Fig. 16

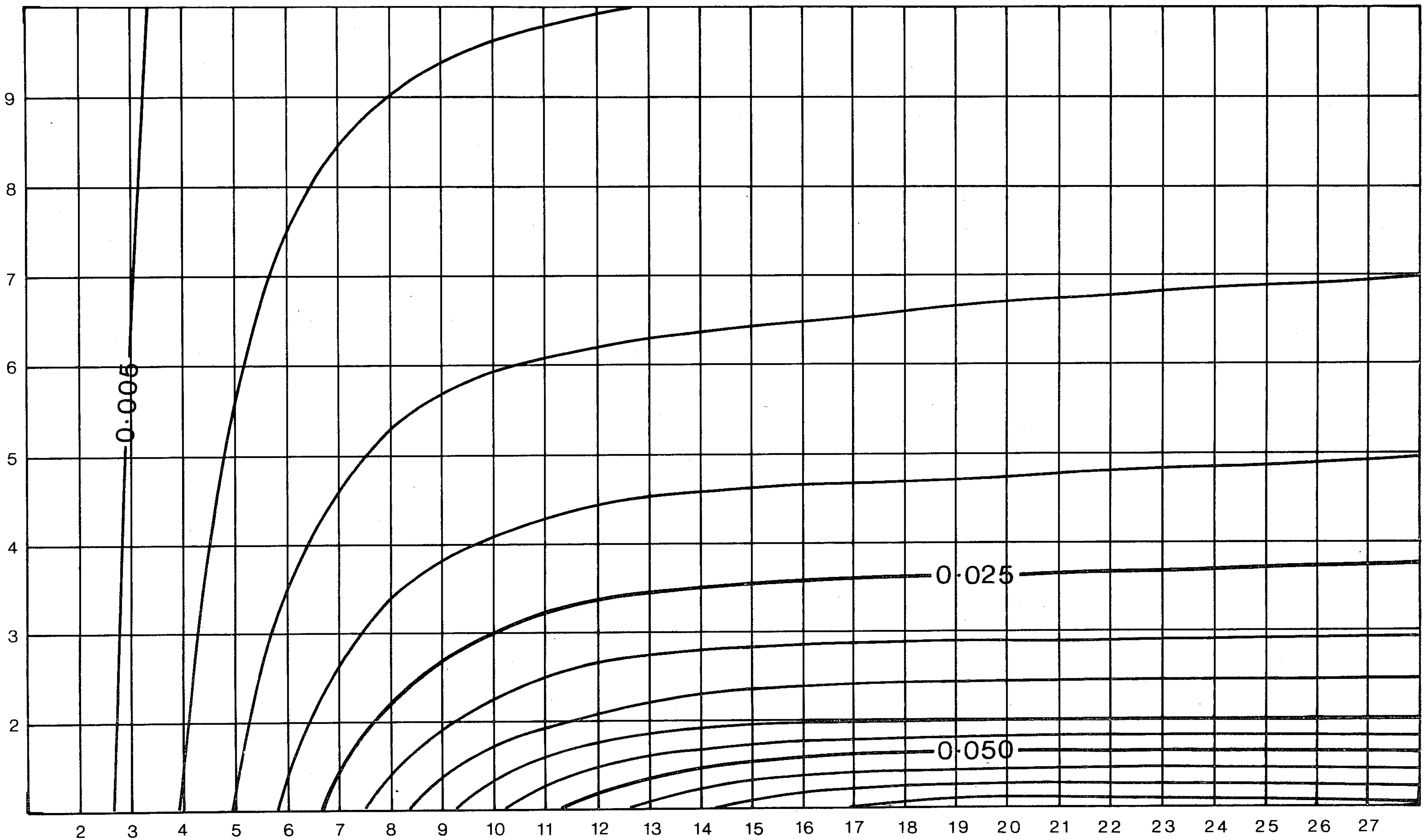


Fig.19a The ratio $\Phi_1/\Phi_o$ as a function of zonal wavelength in $10^6$m and the meridional index n; balanced initial conditions, no initial divergence.

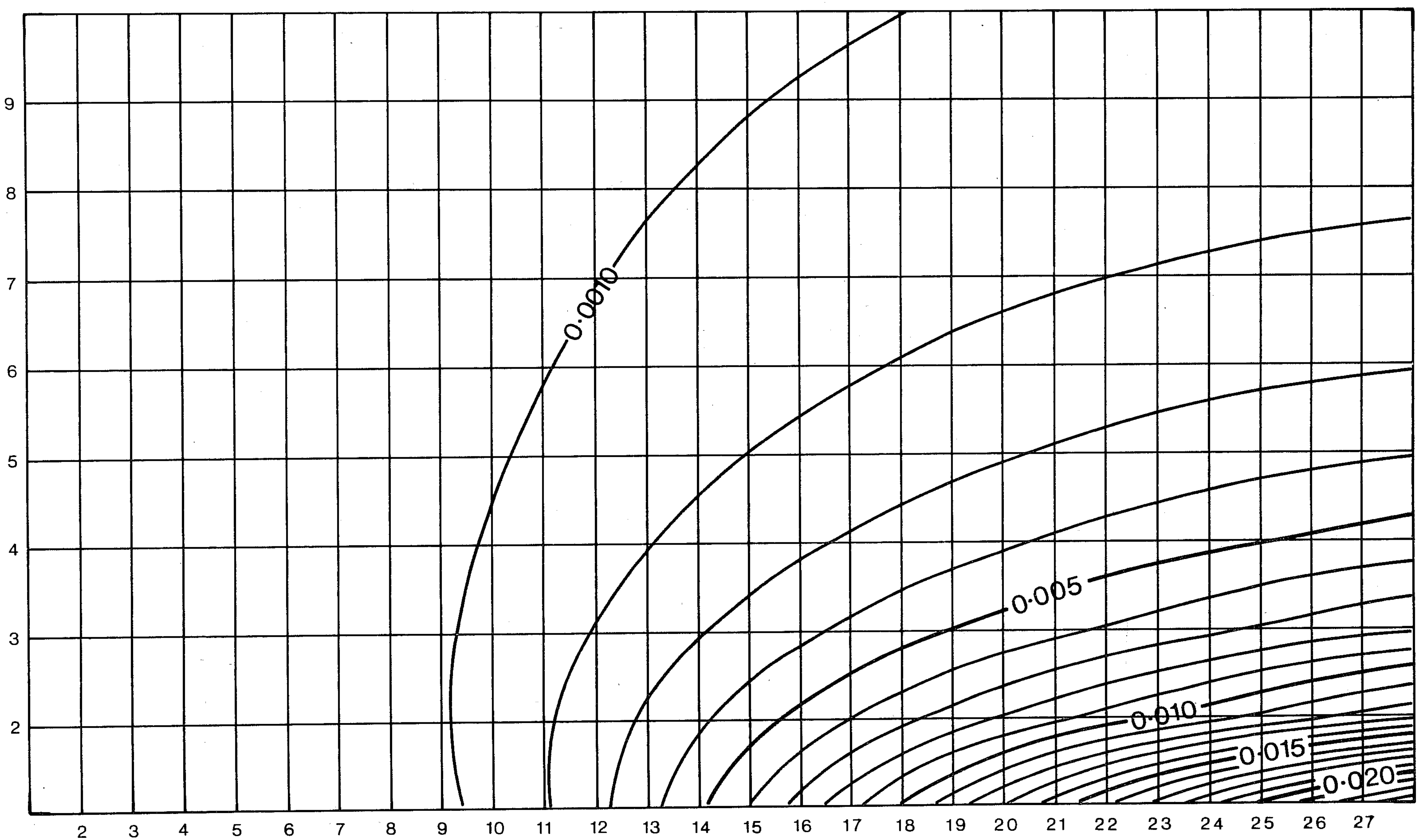


Fig. 19b The ratio $\Phi_1/\Phi_o$ ; initial divergence from quasi-balanced model.

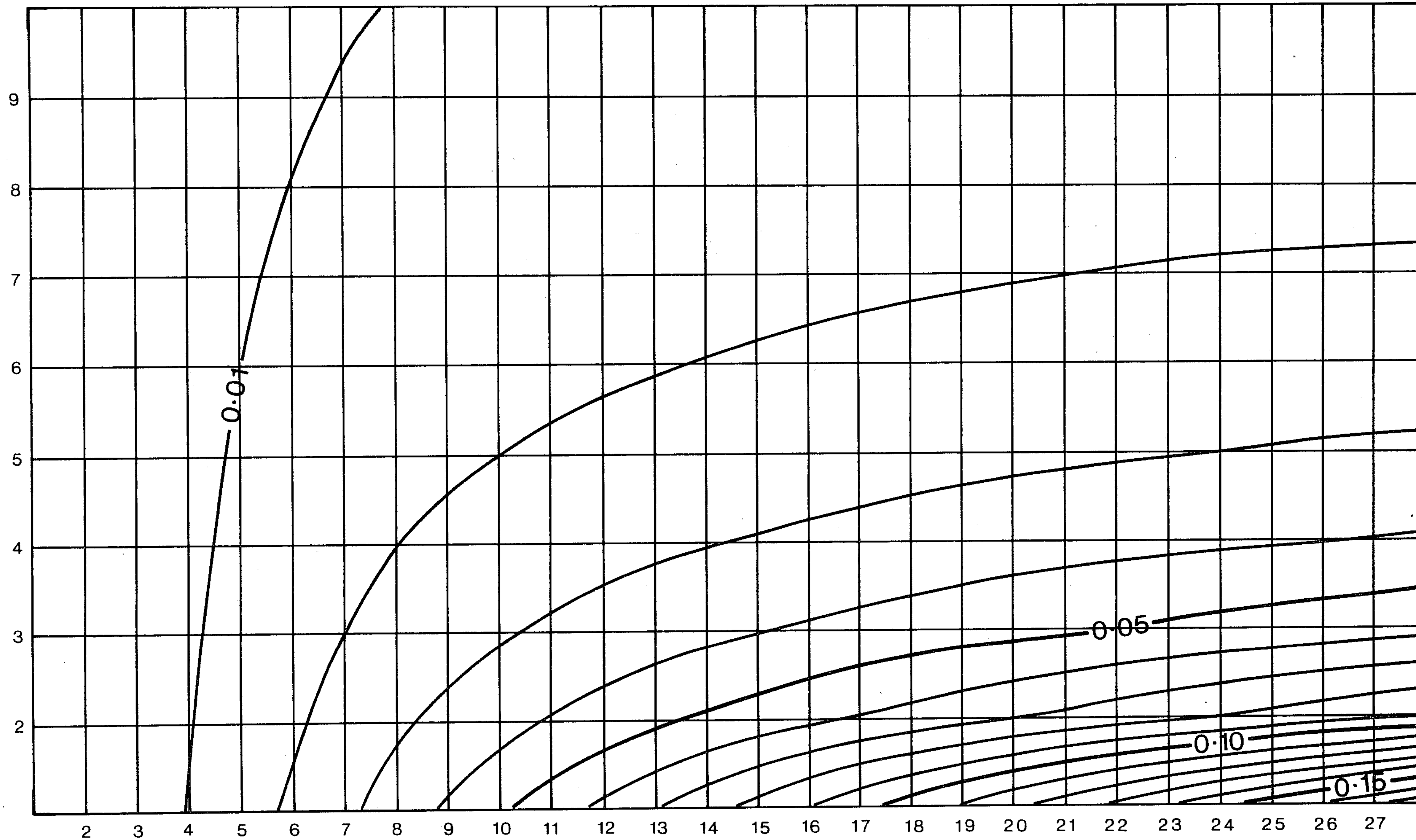


Fig. 20a The ratio $\Phi_2/\Phi_o$; balanced initial conditions, no initial divergence.

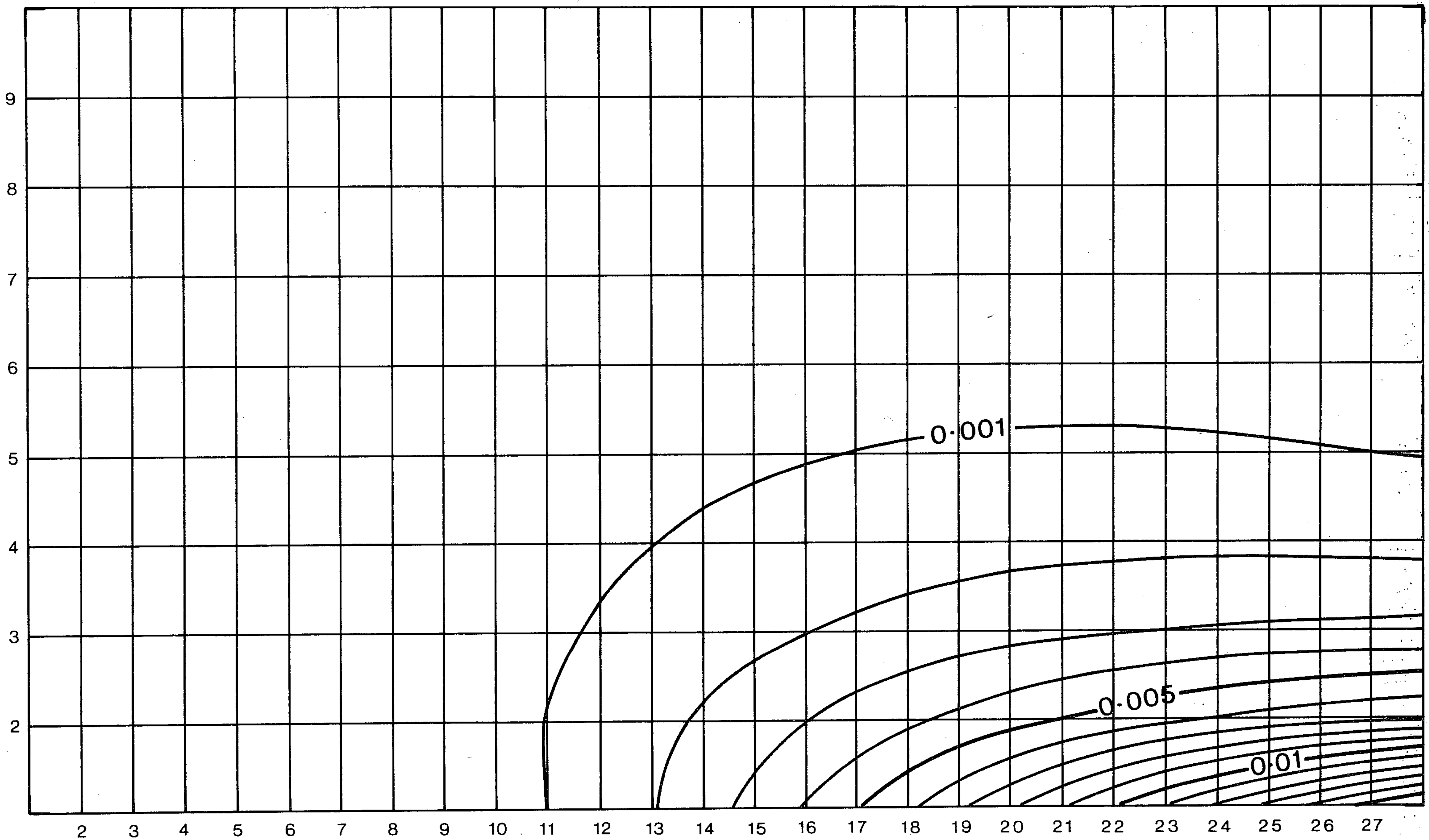


Fig. 20b The ratio $\Phi_2/\Phi_o$; initial divergence from quasi-balanced model.

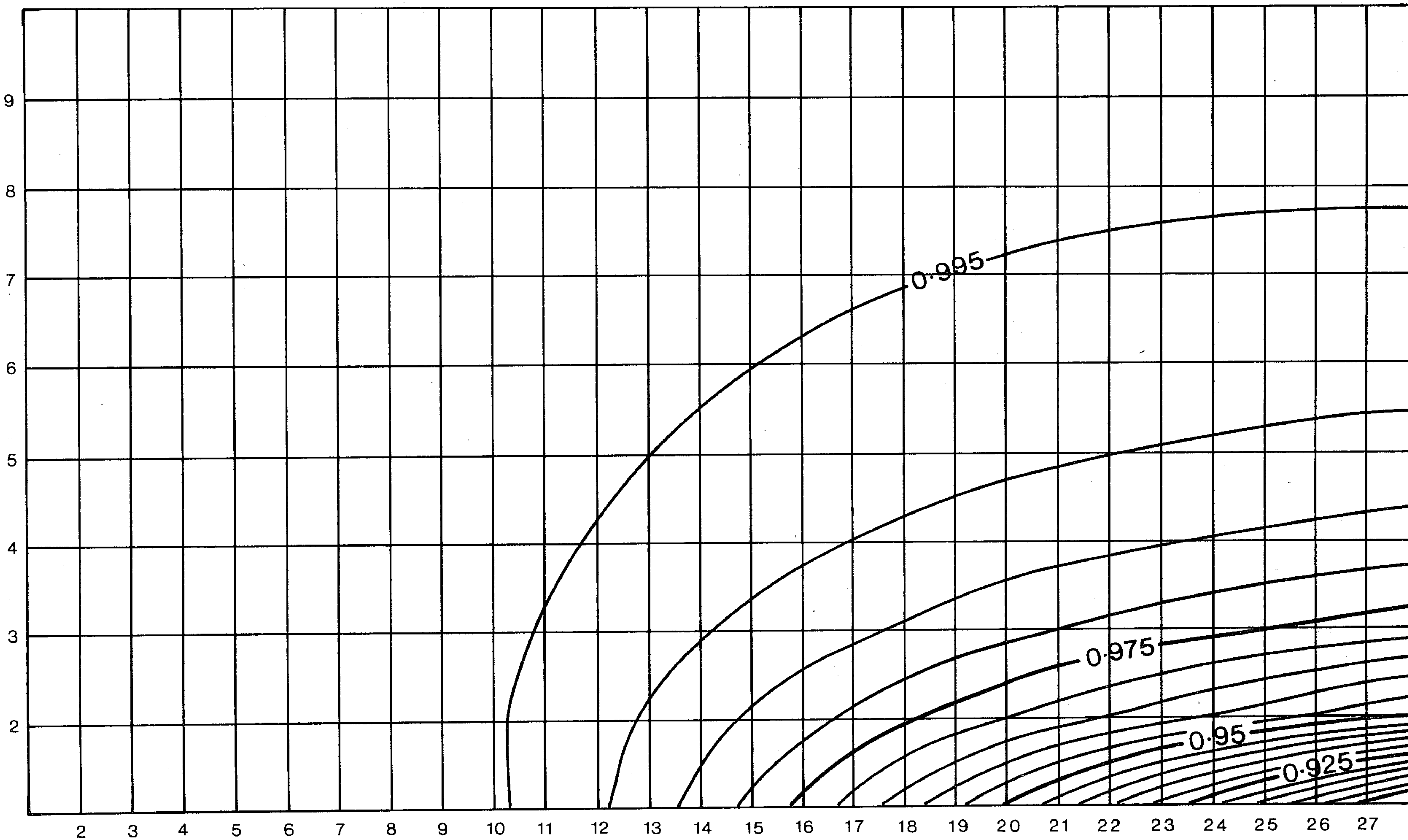


Fig. 21a The ratio $\Phi_3/\Phi_o$; balanced initial conditions, no initial divergence.

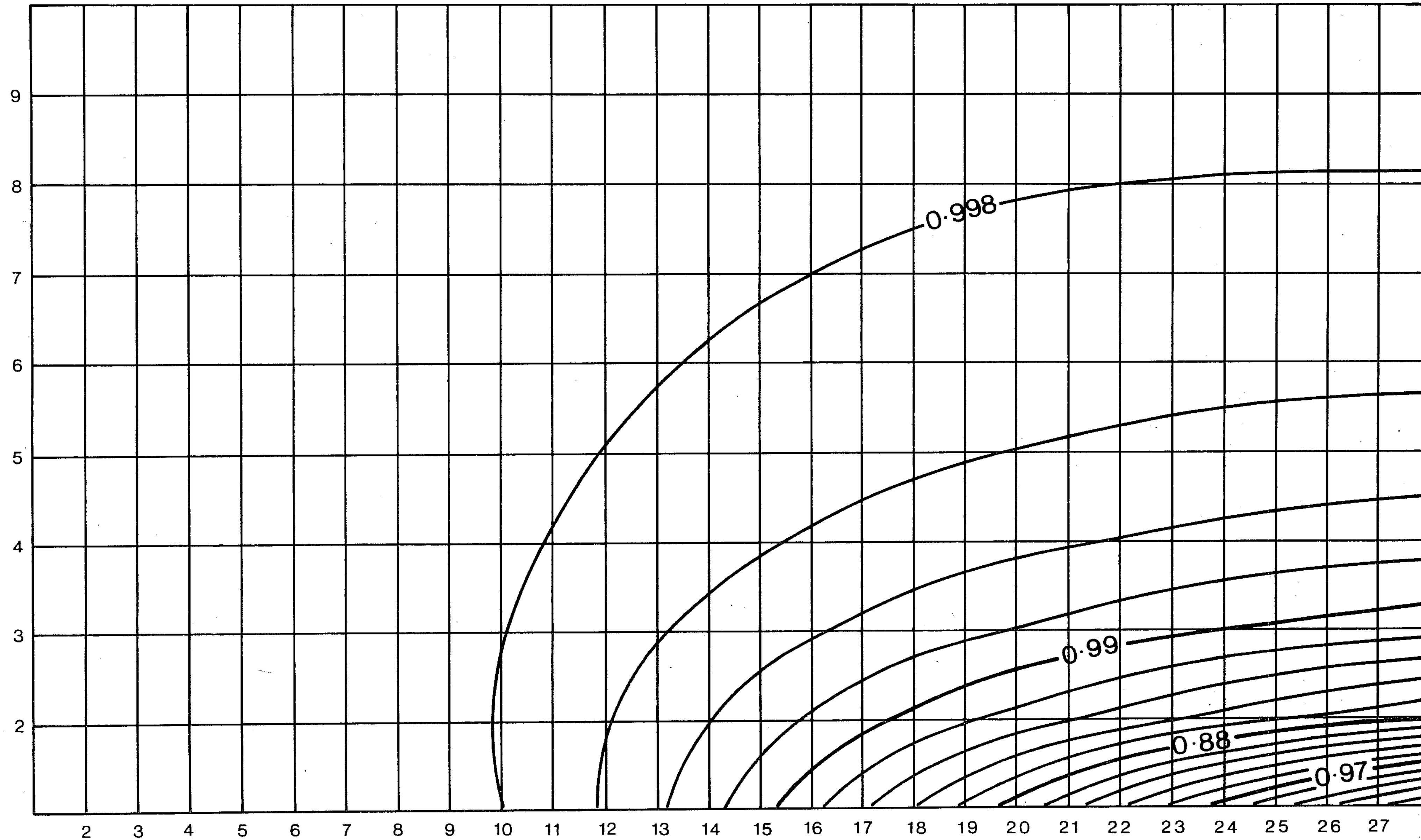


Fig. 21b The ratio $\Phi_3/\Phi_o$; initial divergence from quasi-balanced model.

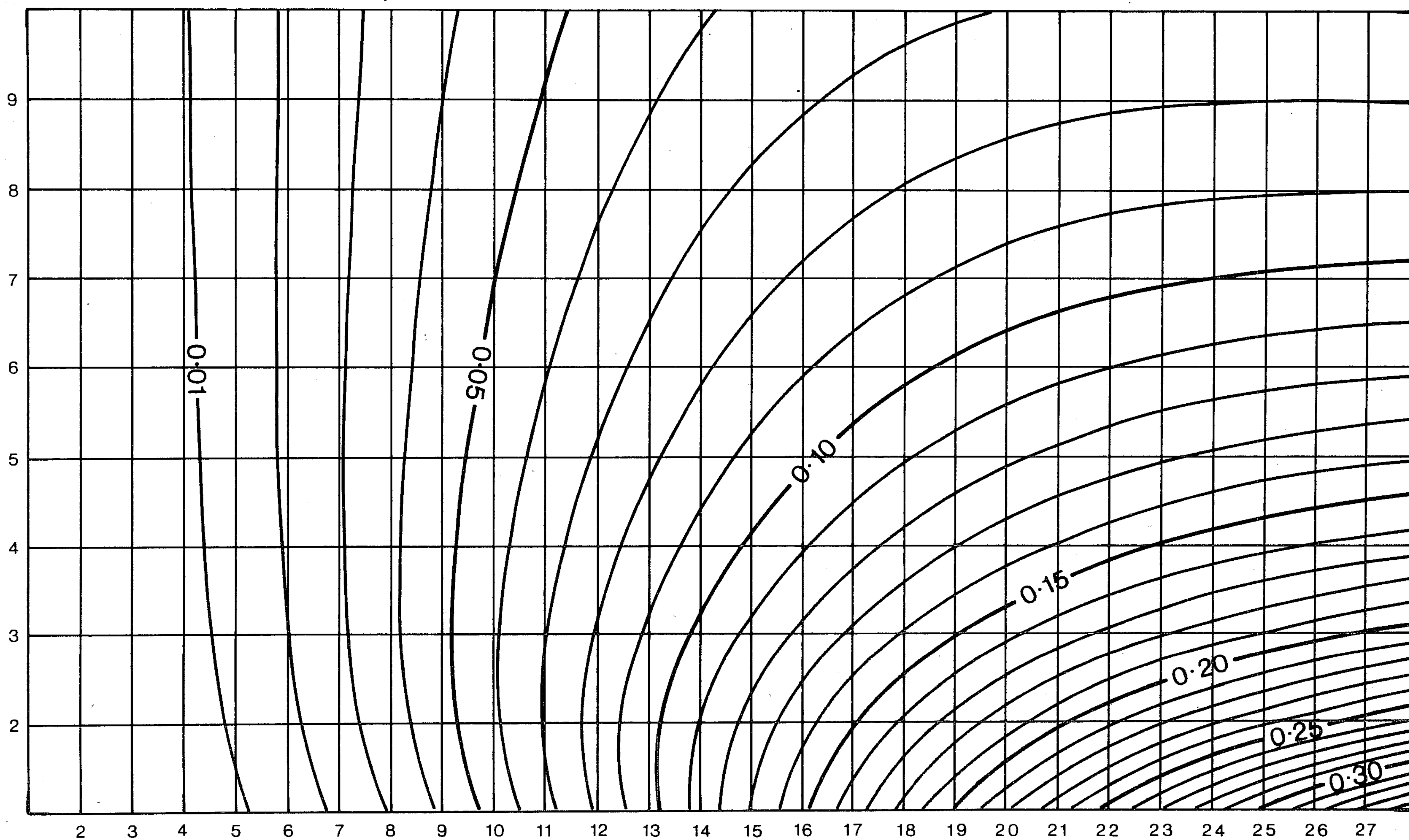


Fig. 22 The ratio $X_3/S_3$ for both initial conditions.

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