B O U N D A R Y L A Y E R S
IN ROTATING FLUIDS

B Y

R. H I D E

METEOROLOGICAL OFFICE

UNITED KINGDOM

Preface

The inviscid equations usually suffice for the description of the flow nearly everywhere in a fluid of low viscosity. Viscous boundary layers arise because the inviscid equations do not satisfy certain constraints which are known to govern the behaviour of real fluids, such as the requirement that the fluid must not slip at a rigid surface or that the stress must be continuous at a free surface. Some boundary layers are "passive" in the sense that they are required merely to "patch up" the solution to the inviscid equations describing the "interior" flow in the main body of the fluid without affecting the "interior" solution; "active" boundary layers, as exemplified in the case of rapidly rotating fluids by the familiar Ekman layer, exert a strong and even dominant influence on the interior solution.

A fluid differs in an essential way from a solid in its inability in the absence of rotation (or of buoyancy forces due to the action of gravity on stable density stratification or, in the case of electrically-conducting fluids, magnetohydrodynamic effects) to resist shearing stresses and thereby support shear waves. When a fluid rotates relative to an inertial frame the constraints imposed on the system by angular momentum requirements are such as to endow the fluid with pseudo-elastic properties which are highly anisotropic, and many novel phenomena then arise. The systematic study of rotating fluids is a developing branch of fluid mechanics with applications not only in meteorology and oceanography but also in other areas of geophysics, planetary sciences, solar physics, astrophysics and engineering. These supplementary notes to two lectures on boundary layers in rotating fluids - which will be presented as a contribution to "Seminars on the treatment of the boundary layer in numerical weather prediction" to be held on 6-10 September 1976 at Shinfield Park, near Reading, organised and sponsored by the European Centre for Medium Range Weather Forecasts - consist of a brief review of the basic properties of geostrophic motion (para.1) , an outline of the properties of Ekman-type boundary layers in a fluid of uniform viscosity and of the more complex (but usually "passive")"side-wall" boundary layers found where the bounding surface is parallel to the basic rotation vector (para.2), and a discussion of the behaviour of certain very simple systems in which these various properties are clearly exemplified (para.3). Time will not permit the treatment of more than a very limited selection of topics, but useful references are listed in the extensive (but unedited and incomplete) bibliography given in Appendix A.

1. INTRODUCTION

1.1 Equations of motion of an incompressible Boussinesq fluid. When dealing with most geophysical and laboratory systems it is sufficient to consider the behaviour of a fluid in which (a) the velocities are so small in comparison with the speed of sound that the assumption of incompressibility is valid, and (b) the accelerations are so small in comparison with gravity that the Boussinesq approximation (which takes density variations into account in the buoyancy term in the equations of motion but not in the other terms) can be used. When referred to a system that rotates with steady angular velocity $\boldsymbol{\Omega}$ relative to an inertial frame, the equations of continuity and momentum of such a fluid of uniform kinematic viscosity ν and variable density $\bar{\rho}(1+\theta)$, where $\bar{\rho}$ is the mean density and $\theta \ll 1$, are:

$$\nabla \cdot \mathbf{u} = 0 \tag{1.1}$$

and

$$\frac{\partial \mathbf{u}}{\partial t} + (2\boldsymbol{\Omega} + \boldsymbol{\xi}) \times \mathbf{u} = -\nabla\left(P + \tfrac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) + \mathbf{g}\theta + \nu\nabla^2\mathbf{u}. \tag{1.2}$$

Here $\mathbf{u}$ is the Eulerian flow velocity relative to the rotating frame and $\boldsymbol{\xi} \equiv \nabla \times \mathbf{u}$ is the corresponding vorticity vector, t denotes time, $\mathbf{g}$ the acceleration due to gravity plus centrifugal effects, and $\bar{\rho}\nabla P$ is equal to the pressure gradient minus $\mathbf{g}\bar{\rho}$.

Variations in density may be due to changes in temperature, salinity, etc., and in general θ satisfies an equation of the form

$$\frac{\partial \theta}{\partial t} + (\mathbf{u}\cdot\nabla)\theta = \chi\nabla^2\theta + \tilde{Q} \tag{1.3}$$

where χ is a diffusion coefficient and $\tilde{Q}$ represents effects due to internal sources in the case of thermally-driven flows $\tilde{Q}$ is proportional to the rate of internal heating per unit mass. When the right-hand side of equation (1.3) vanishes we have

$$\frac{D\theta}{Dt} = 0 \tag{1.4}$$

(where $D/Dt \equiv \partial/\partial t + \mathbf{u}\cdot\nabla$), implying that the value of θ of an individual fluid element then remains constant throughout the motion.

1.2 Energy equation. An energy equation follows from equation (1.2) when that equation is multiplied scalarly by $\mathbf{u}$ (noting that the second term on the left-hand side vanishes because it represents a force acting at right-angles to $\mathbf{u}$ and therefore does no work); whence

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) = -\nu\,\boldsymbol{\xi}\cdot\boldsymbol{\xi} + \mathbf{u}\cdot\mathbf{g}\,\theta - \nabla\cdot\left[\tfrac{1}{2}\mathbf{u}(\mathbf{u}\cdot\mathbf{u}) + \mathbf{u}P + \nu(\boldsymbol{\xi}\times\mathbf{u})\right] \tag{1.5}$$

When integrated over a given volume, the left-hand side represents the rate of change of total kinetic energy and the first term on the right-hand side (which is essentially negative) represents viscous dissipation of kinetic energy. The second term on the right-hand side represents the rate at which buoyancy forces convert into kinetic energy the potential energy of gravity acting on the density field. It can in general take either sign, depending on the sign of the average correlation between density variations, proportional to θ, and the vertical component of velocity, proportional to $\underset{\sim}{u} \cdot \underset{\sim}{g}$, but, in the case of thermally-driven motions, such as the circulation of the atmosphere, this buoyancy term is essentially positive when integrated over the whole system.

The last term on the right-hand side represents mechanical forcing. When integrated, this term can be converted into a surface integral comprising three contributions representing, respectively, the advection of kinetic energy over the surface, the rate of working of normal pressure forces, and the rate of working of tangential viscous forces. Each contribution can take either sign but their sum when integrated over the whole system must be positive in cases of mechanically-driven flows.

1.3 Vorticity equation: Jeffreys's theorem and Ertel's theorem. Equation (1.2) expresses the balance of forces acting on individual fluid elements. The corresponding torque balance is expressed by the vorticity equation obtained by taking the curl of equation (1.2): thus

$$\frac{\partial \underset{\sim}{\xi}}{\partial t} + (\underset{\sim}{u} \cdot \nabla)\underset{\sim}{\xi} - \left[(2\underset{\sim}{\Omega} + \underset{\sim}{\xi}) \cdot \nabla\right]\underset{\sim}{u} = -\underset{\sim}{g} \times \nabla\theta + \nu\nabla^2\underset{\sim}{\xi} \,. \qquad (1.6)$$

This equation leads directly to a general result which goes under several names but is conveniently referred to as "Jeffreys's theorem" and concerns the conditions under which hydrostatic equilibrium obtains, defined as $\underset{\sim}{u} = 0$ everywhere. By equation (1.6), $\underset{\sim}{u} = 0$ when $\underset{\sim}{g} \times \nabla\theta = 0$, implying that hydrostatic equilibrium is impossible if density variations occur on level surfaces. Jeffreys's theorem is a direct corollorary of Bjerknes's well-known circulation theorem; it provides the most direct demonstration that the atmosphere must circulate under the influence of solar heating, which maintains a generally north-south density gradient on level surfaces.

We now introduce a quantity known as "potential vorticity" and defined as

$$(2\underset{\sim}{\Omega} + \underset{\sim}{\xi}) \cdot \nabla\Lambda \qquad (1.7)$$

where Λ is any quantity satisfying

$$\frac{D\Lambda}{Dt} = 0 \qquad (1.8)$$

(of equation (1.4)). By equation (1.6)

$$\frac{D}{Dt}\left[(2\underset{\sim}{\Omega} + \underset{\sim}{\xi}) \cdot \nabla\Lambda\right] = -(\underset{\sim}{g} \times \nabla\theta) \cdot \nabla\Lambda + \nu\nabla\Lambda \cdot \nabla^2\underset{\sim}{\xi}$$

and therefore

$$\frac{D}{Dt}\left[(2\Omega + \xi).\nabla\Lambda\right] = 0 \qquad (1.10)$$

when the fluid is homogeneous $(\nabla\theta = 0)$ and inviscid. This is Ertel's particularly useful theorem for an incompressible Boussinesq fluid.

1.4 Geostrophic flow; thermal wind equation and Proudman's theorem. Geostrophic flow occurs in regions where the relative acceleration term $Du/Dt \; (= \partial u/\partial t + (\xi \times u) + \nabla(\frac{1}{2} u.u))$ in equation (1.2) and the viscous term $\nu\nabla^2 u$ can be neglected in comparison with the Coriolis term $2\Omega \times u$. The Coriolis force then balances the non-hydrostatic component of the pressure force exactly, so that

$$\boxed{2\Omega \times u = -\nabla P + g\theta.} \qquad (1.11)$$

This equation is mathematically degenerate, being of lower order than the complete equation of motion and consequently incapable of solution under all the necessary boundary conditions and initial conditions, and it follows that regions of highly ageostrophic flow (occurring not only on the boundaries of the system but also in the localized regions of the main body of the fluid) are necessary concomitants of geostrophic motion. The geostrophic equation nevertheless expresses with good accuracy various important properties that slow, steady hydrodynamical motions in a rapidly-rotating fluid must possess nearly everywhere, and when judiciously applied the equation usually indicates the nature and location of essentially ageostrophic features.

A rapidly rotating fluid can be defined as one for which the Rossby number

$$\epsilon \equiv \langle Du/Dt \rangle / \langle 2\Omega \times u \rangle \qquad (1.12)$$

and the Ekman number

$$E \equiv \langle \nu\nabla^2 u \rangle / \langle 2\Omega \times u \rangle \qquad (1.13)$$

are both very much less than unity, the symbol $\langle\ \rangle$ meaning the root mean square value taken over the whole volume occupied by the fluid, so that $\epsilon = \hat{U}/\hat{L}\Omega$ and $E = \nu/\hat{L}^2\Omega$ if $\hat{U}$ is a typical relative flow speed and $\hat{L}$ a characteristic length scale. From a mathematical point of view, geostrophic flow occurs in the limit when $\epsilon \to 0$ and $E \to 0$. The vorticity equation (1.6) then simplifies to

$$\boxed{(2\Omega.\nabla)u = g \times \nabla\theta} \qquad (1.14)$$

expressing a balance between the gyroscopic torque and the gravitational torque.

When $\nabla\theta = 0$ equation (1.14) reduces to

$$2\Omega \frac{\partial \underset{\sim}{u}}{\partial z} = 0 \qquad (1.15)$$

(where the z axis is parallel to $\underset{\sim}{\Omega}$),
a result first proved by Proudman and later by others and which goes under various names (eg Proudman's theorem, Proudman-Taylor theorem, Taylor-Proudman theorem). In words, Proudman's "two-dimensional" theorem states that <u>geostrophic motion of a homogeneous fluid will be the same in all planes perpendicular to the axis of rotation.</u> This fundamental result underlies the interpretation of a very wide range of phenomena in mechanically-driven flow systems.

Suppose that (U,V,W) are the (X,Y,Z) components of $\underset{\sim}{u}$, where Z is the downward vertical co-ordinate, so that in these co-ordinates $\underset{\sim}{g} = (0,0,g)$, W is the corresponding vertical component of motion, and (U,V) are the horizontal components. When $\nabla\theta \neq 0$ we have, by equation (1.14),

$$\left(2\underset{\sim}{\Omega}.\nabla\right)(U,V,W) = g\left(-\partial\theta/\partial Y, \partial\theta/\partial X, 0\right). \qquad (1.16)$$

In cases when the horizontal component of $\underset{\sim}{\Omega}$ is negligible, the first two components of equation (1.16) comprise the familiar thermal wind equation, which expresses the relationship between the vertical rate of change of horizontal geostrophic motion and the horizontal density gradient. It may be shown by combining equation (1.16) with equation (1.4) and setting $\partial\theta/\partial t = 0$ that under steady isentropic conditions

$$\left(2\underset{\sim}{\Omega}.\nabla\right)(V/U) = -\left(g\,W\partial\theta/\partial Z\right)/(U^2+V^2), \qquad (1.17)$$

implying that when, as a result of strong density inhomogeneities, the <u>speed</u> of horizontal flow varies rapidly with respect to the axial co-ordinate z, the corresponding rate of change of the <u>direction</u> of horizontal flow may be quite slow and even vanish altogether when $W\partial\theta/\partial Z = 0$.

1.5 <u>Quasi-geostrophic flow of an inviscid fluid.</u> Quasi-geostrophic flow occurs when $E \ll 1$ and $\epsilon \ll 1$, and if $E \ll \epsilon$ the dominant ageostrophic contributions in the equations of quasi-geostrophic motion are provided by advective effects, not viscosity. Thus

$$\frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u}_1.\nabla_1)\underset{\sim}{u} + 2\underset{\sim}{\Omega}\times\underset{\sim}{u} \doteqdot -\nabla P + \underset{\sim}{g}\,\theta \qquad (1.18)$$

where $\underset{\sim}{u}_1.\nabla_1 \equiv u\,\partial/\partial x + v\,\partial/\partial y$, and the corresponding equation for ζ (where $\underset{\sim}{\xi} = (\xi,\eta,\zeta)$) is

$$\frac{\partial \zeta}{\partial t} + \left(\underset{\sim}{u}_1.\nabla_1\right)\zeta \doteqdot 2\Omega\frac{\partial w}{\partial z} - \left(\underset{\sim}{g}\times\nabla\theta\right)_z \qquad (1.19)$$

(see equations (1.2), (1.6), (1.12) and (1.13)). Equation (1.19) shows that in quasi-geostrophic motion of a homogeneous (ie $(g\times\nabla\theta)_z = 0$) incompressible fluid, changes in the relative vorticity of a moving fluid element are brought about largely by axial stretching, as represented by the term $2\Omega\, \partial w/\partial z$ on the right-hand side.

Suppose for the moment that the buoyancy term can be neglected and that the fluid is bounded by rigid end-walls in $z = z_\ell(x, y)$ and $z = z_u(x, y)$ where $z_\ell < z_u$. When effects due to viscous boundary layers are negligible the term $2\Omega\partial w/\partial z$ equals $2\Omega(\underset{\sim}{u}_1.\nabla_1)\ln(z_u - z_\ell)$ (to sufficient accuracy) and the equation (1.19) reduces to an expression for the conservation of potential vorticity $(2\Omega + \zeta)/(z_u - z_\ell)$, namely

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{u}_1.\nabla_1\right)\left(\frac{2\Omega + \zeta}{z_u - z_\ell}\right) = 0. \qquad (1.20)$$

(Equation (1.20) follows directly from Ertel's potential vorticity theorem given by equation (1.10) when $\underset{\sim}{u}_0.\nabla$ is approximated by its transverse part $\underset{\sim}{u}_1.\nabla_1$, and $\Lambda = (z - z_\ell)/(z_u - z_\ell)$ or $\Lambda = (z_u - z)/(z_u - z_\ell)$.)

If θ satisfies equation (1.4) we can set $\Lambda = \theta$ (of equation (1.8)) in equation (1.9), and if we further assume that $\nu = 0$ the right-hand side vanishes, giving

$$\frac{D}{Dt}\left[(2\underset{\sim}{\Omega} + \underset{\sim}{\xi}).\nabla\theta\right] = 0. \qquad (1.21)$$

In the geostrophic limit, this potential vorticity equation for a non-homogeneous fluid has no general form analogous to equation (1.20), but for a shallow system, such as the Earth's atmosphere, in which $\epsilon \ll 1$ (but $> E$), $\Theta = \Theta_0(Z) + \delta\theta$ with $\delta\theta \ll \Theta_0$, $P = P_0(Z) + \delta P$ with $\delta P \ll P_0$ and f is the vertical component of 2Ω, so that by equation (1.11)

$$(\underset{\sim}{\xi})_Z = -f^{-1}\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right)\delta P \quad \text{and} \quad \delta\theta = g^{-1}\frac{\partial\,\delta P}{\partial Z} \qquad (1.22)$$

equation (1.21) reduces to

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial X} + V\frac{\partial}{\partial Y}\right)\left\{\left(\frac{\partial\Theta_0}{\partial Z}\right)\left[f + \frac{1}{f}\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{f^2\partial^2/\partial Z^2}{g\,\partial\Theta_0/\partial Z}\right)\right]\delta P\right\} = 0. \qquad (1.23)$$

Equation (1.23) is of central importance in a wide range of theoretical investigations in dynamical meteorology and oceanography, including the study of "geostrophic turbulence" where the constraints on potential vorticity represented by equation (1.23) or (1.20) place severe restrictions on the types of non-linear interactions that are possible. This results in behaviour which is analogous in some respects to that of two-dimensional turbulence in a homogeneous fluid.

2. BOUNDARY LAYERS

2.1 Ekman-type boundary layers. Boundary layers arise because the inviscid equations do not satisfy certain constraints which are known to govern the behaviour of real fluids. These are, for instance, that the fluid must not slip at a rigid surface, or that the stress must be continuous at a free surface. When the inviscid equations are those satisfied by geostrophic motion (see equation (1.11)) and the bounding surface is not parallel to $\underset{\sim}{\Omega}$ the boundary layer is of the so-called Ekman type of thickness δ, where

$$\delta = \left(\frac{\nu}{|\underset{\sim}{\Omega}\cdot\underset{\sim}{n}|}\right)^{\frac{1}{2}}\left(1+O(\epsilon)\right) \tag{2.1}$$

(see equation (1.12)) if $\underset{\sim}{n}$ is a unit vector normal to the bounding surface. In the case of a rigid surface, the components (u_1, u_2) of $\underset{\sim}{u}$ parallel to the wall are given by

$$u_1 = \left\{U_1\left(1-e^{-\sigma}\cos\sigma\right) - \mathrm{sgn}(\underset{\sim}{n}\cdot\underset{\sim}{\Omega})\,U_2\, e^{-\sigma}\sin\sigma\right\}\left(1+O(\epsilon)\right) \tag{2.2}$$

and

$$u_2 = \left\{U_2\left(1-e^{-\sigma}\cos\sigma\right) + \mathrm{sgn}(\underset{\sim}{n}\cdot\underset{\sim}{\Omega})\,U_1\, e^{-\sigma}\sin\sigma\right\}\left(1+O(\epsilon)\right) \tag{2.3}$$

where (U_1, U_2) are the corresponding components at the edge of the boundary layer, where it meets the geostrophic "interior" region, σ is a stretched co-ordinate equal to the distance x_3 from the bounding surface divided by δ. It is clear from equation (1.5) that viscous dissipation of energy occurs largely within the boundary layer, at a rate $\propto \rho\nu \hat{U}^2/\delta^2 \sim \Omega\rho\hat{U}^2$ per unit volume. ¶ A net cross-isobar flow takes place in the boundary layer and if the x_3 - component of the vorticity in the interior region, $\partial U_2/\partial x_1 - \partial U_1/\partial x_2$, is non-zero, boundary layer "suction" occurs, giving rise to a value of u_3 (the component of $\underset{\sim}{u}$ normal to the boundary) which vanishes on the rigid boundary only and has the generally non-zero value

$$\boxed{U_3 = \mathrm{sgn}(\underset{\sim}{\Omega}\cdot\underset{\sim}{n})\,\frac{\delta}{2}\left(\frac{\partial U_2}{\partial x_1} - \frac{\partial U_1}{\partial x_2}\right)} \tag{2.4*}$$

at the edge of the boundary layer (see e.g. Batchelor 1967, Greenspan 1968, Prandtl 1952 in list of references given in Appendix A).

* Footnote: When δ is a function of position another term, proportional to $U_2\,\partial\delta/\partial x_1 - U_1\,\partial\delta/\partial x_2$ must be included on the right hand side

When Ekman boundary layer suction occurs the factor $\partial w/\partial z$ of the first term on the right-hand side of equation (1.19) does not vanish even in the absence of topographic stretching, <u>implying that the boundary layer can exert a direct influence on the axial vorticity of the interior flow.</u> In the case of a homogeneous fluid bounded by rigid surfaces in $z = z_u(x,y)$ and $z = z_\ell(x,y)$, $z_u > z_\ell$, by equations (1.15), (1.19) and (2.4), the axial component of relative vorticity of the interior flow satisfies

$$\frac{\partial \zeta}{\partial t} + (\underset{\sim}{u}_1 \cdot \nabla_1)\zeta \doteqdot \frac{2\Omega}{D}\left[(\underset{\sim}{u}_1 \cdot \nabla_1) D - \delta \zeta\right] \tag{2.5}$$

where $D = z_u - z_\ell$ and $\underset{\sim}{u}_1 \cdot \nabla_1 D = u\partial D/\partial x + v\partial D/\partial y$. When topographic and advective effects are negligible (i.e. $\underset{\sim}{u}_1 \cdot \nabla_1 D$ is typically much smaller in magnitude than $\delta\zeta$ and $(\underset{\sim}{u}_1 \cdot \nabla_1)\zeta$ smaller than $\partial\zeta/\partial t$) we have $\partial\zeta/\partial t \doteqdot -2\Omega\delta\zeta/D$, implying that ζ decreases exponentially with a decay time

$$D/2\Omega\delta, \tag{2.6}$$

the so-called "spin-up" time scale. The bibliography in Appendix A contains many references to work on the spin-up process in both homogeneous and stably-stratified fluids, including reviews by Greenspan (1968) and Benton and Clark (1974).

In the case of a non-homogeneous fluid in which, owing to the presence of horizontal density gradients, the geostrophic interior flow varies with z according to the thermal wind equation (see equations (1.14) and (1.16)), it is still possible over a wide range of conditions to apply as boundary conditions the Ekman suction formula given by equation (2.4) as, for example, in the case of Barcilon's (1964) investigation of the influence of Ekman boundary layers at rigid end-walls on the process of baroclinic instability. At a <u>free</u> surface in a baroclinic fluid, continuity of stress requires the presence of a boundary layer in which $\partial u_1/\partial x_3$ and $\partial u_2/\partial x_3$ undergo an Ekman spiral (Hide 1964), the concomitant suction formula being

$$U_3 = -\operatorname{sgn}(\underset{\sim}{\Omega}\cdot\underset{\sim}{n})\frac{\delta^2}{2}\frac{\partial}{\partial x_3}\left(\frac{\partial U_2}{\partial x_1} - \frac{\partial U_1}{\partial x_2}\right) \tag{2.7}$$

(cf. equation (2.4)). In certain meteorological problems (see e.g. Charney 1969) it is appropriate to regard the tropopause as a free surface and apply to the interior geostrophic flow a boundary condition based on equation (2.7). (For further references and applications see Busse 1968, Charney 1973, Hide 1969, Hide and Mason 1975).

2.2 <u>Side-wall boundary layers and detached shear layers.</u> The Ekman-thickness δ becomes infinite when $\underset{\sim}{\Omega}\cdot\underset{\sim}{n} = 0$ (see equation (2.1)), so that Ekman theory breaks down when dealing with bounding surfaces parallel to $\underset{\sim}{\Omega}$. The mathematical analysis of these so-called "side-wall" boundary layers is highly complex, even in the case when non-linear advective effects are negligible and the layers are consequently of the Stewartson type (see e.g. Greenspan 1968, Brown and Stewartson 1976), with overall thickness.

$$\sim (D\delta/2)^{1/2} \tag{2.8}$$

(and therefore proportional to $\nu^{1/4}$) and substructure on the scale $D^{1/3}\delta^{2/3}$ (proportional to $\nu^{1/3}$).

In the Stewartson-type boundary layer, both terms on the left-hand side and the topographic term on the right-hand side of the equation (2.5) are negligible. The vorticity balance is then between the Ekman suction term $-2\Omega\delta\zeta/D$ and the horizontal diffusion term $\nu\nabla^2\zeta$, which can no longer be ignored (see equation (1.6)), as we shall see in various examples to be considered later. Equation (2.5) shows that the error in linear theory is $O(L\epsilon/D^{1/2}\delta^{1/2})$ and therefore greater than O(ϵ), implying that non-linear effects must be taken into account in the treatment of side-wall boundary layers even when linear (Ekman) theory suffices for the end-wall boundary layers (see Bennetts and Jackson 1974, Hide 1968).

It is a common observation with an elementary theoretical interpretation (see § 1.4) that highly ageostrophic detached shear layers occur in rapidly rotating fluids and that in many cases these layers are parallel to $\underset{\sim}{\Omega}$ and similar in structure to side-wall boundary layers. So far as the geostrophic interior flow is concerned, however, it is a fortunate circumstance that this flow can usually be determined with the aid of end-wall boundary conditions based on Ekman theory without having first to determine the detailed structure of side-wall boundary layers or detached shear layers parallel to $\underset{\sim}{\Omega}$, provided, of course, that these layers are not subject to shear instability (see e.g. Hide and Titman 1967). Further complications arise when ϵ (though less than unity) is such that a Reynolds number $\hat{U}\delta/\nu$ based on the Ekman thickness exceeds ~ 10 (for a homogeneous fluid) and, in consequence, the Ekman boundary layers are subject to shear instability (see e.g. Faller 1963, Greenspan 1968).

3. SOME EXAMPLES

3.1 <u>Axisymmetric source-sink flow.</u> The simplest conceivable flow in a rotating system is the axisymmetric motion that arises in a fluid bounded by two concentric rigid porous cylinders in r=a and r=b (b > a) when fluid enters and leaves the system <u>via</u> the cylinders at the constant rate $q/2\pi$ m^3 per second per unit length in the z direction per unit angular distance in the azimuthal direction and it can be assumed (a) that the entering fluid has zero vorticity relative to the bounding surfaces, which rotate with the same steady axial velocity $\underset{\sim}{\Omega}$ about the axis of symmetry and (b) that the flow is independent not only of the azimuthal co-ordinate ϕ but also of the axial co-ordinate z . Thus $\underset{\sim}{u}=(u_r,u_\phi,u_z)$ where $u_r=q/2\pi r$, $u_z=0$ and

$$u_\phi=\Omega\left\{-r+\frac{1}{(b^{S+2}-a^{S+2})}\left[\frac{b^2a^2(b^S-a^S)}{r}+(b^2-a^2)r^{S+1}\right]\right\} \tag{3.1}$$

if $S\equiv q/2\pi\nu$ (ν being the coefficient of viscosity) and q is reckoned positive or negative according as the inner cyclinder is the source of sink of fluid. Figure A1 of Hide (1968) illustrates the radial dependence of the profile of u_ϕ on the Reynolds number $|S|$, for several values of S ranging from $-\infty$ to ∞ . When $|S|$ is very small, viscosity ensures that the relative azimuthal motion u_ϕ is very slow, but when $|S|\gg 1$, viscous effects are confined to a boundary layer on the sink of thickness

$$b/|S| \quad \text{or} \quad a/|S| \quad \text{according as} \quad q \gtrless 0. \tag{3.2}$$

The azimuthal flow elsewhere is such that individual fluid elements conserve their angular momentum, so that $\zeta \equiv r^{-1}\partial(r u_\phi)/\partial r$, the axial and only non-zero component of relative vorticity $\underset{\sim}{\xi}$, is equal to -2Ω ; this can be seen from the general expression

$$\underset{\sim}{\xi} = (0,0,\zeta) = \left(0,0,2\Omega\left\{\left(1+\tfrac{1}{2}S\right)\left[\frac{(b^2-a^2)r^S}{b^{S+2}-a^{S+2}}\right]-1\right\}\right) \tag{3.3}$$

The corresponding absolute vorticity in the main body of the fluid is zero, implying — since the area integral of the absolute vorticity can be shown to equal $2\pi(b^2-a^2)\Omega$ for all S — that when $|S| \gg 1$ the absolute vorticity is concentrated in the thin boundary layer on the surface where the fluid leaves the system. (This is a clear case of motions expelling absolute vorticity from the main body of the fluid and concentrating it at the rim but by a process which can be fully specified, in contrast to some of the examples invoked during the controversy — still apparently unsettled — started in the 1960's by certain speculations concerning the early stages in the development of hurricanes.)

<u>Strictly</u> two-dimensional flows are impossible to realise in practice, owing to the presence of <u>end-walls</u> in $z = z_\ell$ and $z = z_u$ (where $z_u > z_\ell$). The "end effects" produced by such walls range in general from minor local modifications when the basic two-dimensional flow has no relative vorticity and the end-walls are everywhere perpendicular to $\underset{\sim}{\Omega}$ (ie when $\nabla_1 z_\ell = \nabla_1 z_u = 0$), to major changes in the flow pattern throughout the whole system when the basic two-dimensional flow (such as the one we are now considering) possesses vorticity, or the end-walls are not everywhere perpendicular to $\underset{\sim}{\Omega}$. Consider the case when the end-walls are perpendicular to $\underset{\sim}{\Omega}$, so that $\nabla_1 z_u = \nabla_1 z_\ell = 0$ and the separation distance, in general given by

$$D \equiv z_u(r,\phi) - z_\ell(r,\phi), \tag{3.4}$$

is uniform (see Figure 4 of Hide 1968). Suppose that the coefficient of viscosity is so small that boundary layers of thickness much less than D develop on each end-wall. When Ω is sufficiently large the interior flow is quasi-geostrophic (ie $\epsilon \ll 1$ and $E \ll 1$, see equations (1.12) and (1.13)) and the end-wall boundary layers are of the Ekman type (see equations (2.2), (2.3) and (2.4)).

The flow can now be divided into five regions, namely the inviscid "interior" region where the flow is quasi-geostrophic, satisfying

$$\underset{\sim}{u} = (u_r, u_\phi, u_z) = \left(0,\ Q\Omega^{\frac{1}{2}}/2\pi\nu^{\frac{1}{2}}r,\ 0\right) \times \left[1 + O(\epsilon)\right] \tag{3.5}$$

where $Q \equiv q D$ and the Rossby number

$$\epsilon = |Q| / 2\pi\nu^{\frac{1}{2}}\Omega^{\frac{1}{2}}a^2 \tag{3.6}$$

(E here being defined as $\nu/\Omega D^2$),

and four highly ageostrophic regions comprising two Ekman layers of thickness $\delta = (\nu/\Omega)^{\frac{1}{2}}$ on the end-walls separated by the uniform distance D and boundary layers of thickness Δ_a and Δ_b on the side-walls in r=a and r=b, supposing that $\Delta_a + \Delta_b \ll$ b-a. The transfer of fluid now takes place *via* these boundary layers but it is theoretically significant (see § 2.2 above) that simple Ekman theory, without recourse to consideration of the complex structure of the side-wall boundary layers, can be used to determine in the inviscid interior flow with an error no more than $O(\epsilon)$. Within that region all components of relative vorticity ξ now vanish (to $O(\epsilon)$)), in contrast to the case when the end-walls are absent (cf. equation (3.3)) since Proudman's theorem (see equation (1.15)) requires that geostrophic flow of a homogeneous fluid should satisfy $\partial \underset{\sim}{u}/\partial z = 0$, the first two components of which when combined with equation (2.4) give

$$\frac{\partial u_z}{\partial z} = -\frac{\delta \zeta}{D}, \tag{3.7}$$

which is only compatible with the third component $\partial u_z/\partial z = 0$ when $\zeta = 0$

The mathematical analysis of the side-wall boundary layers is highly complex, (see § 2.2 above). According to an approximate analysis and a supporting laboratory investigation (Hide 1968), the thickness of the boundary layer on the source (Δ_a when $q > 0$) increases and that of the sink boundary layer (Δ_b when $q > 0$) decreases with increasing $\epsilon E^{-1/4}$ (and vice-versa when $q < 0$), but in such a way that the product $\Delta_a \Delta_b$ remains $\sim D\delta/2$ even when $\epsilon E^{-1/4}$ is quite large, with

$$\Delta_b \doteq 2\pi \nu b D/Q \quad \text{and} \quad \Delta_a \doteq Q/4\pi \nu^{1/2} \Omega^{1/2} a \tag{3.8}$$

when $\epsilon E^{-1/4} \gg 1$, in contrast to the case when $\epsilon E^{-1/4} \ll 1$ and $\Delta_a = \Delta_b = D^{1/2} \nu^{1/4} / 2^{1/2} \Omega^{1/4}$.

These results have been generally confirmed and extended by further work, including a combined numerical and laboratory investigation by Bennetts and Jackson (1974).

Because the flow is axisymmetric, it is a relatively straightforward matter to extend the foregoing analysis to cases when the end-walls are no longer perpendicular to $\underset{\sim}{\Omega}$ provided that in shape they remain figures of revolution about the axis of symmetry, since differences from the case we have just considered are mainly only quantitative. Thus, when the bounding end-wall surfaces are concentric spheres of radii $\hat{a}$ and $\hat{b}$ ($\hat{a} < \hat{b}$) and relative flow is produced by a cylindrical source near one pole feeding a cylindrical sink near the other pole, the transfer of fluid, again, takes place *via* Ekman layers, which now have thickness

$$\delta = (\nu / |\Omega \cos \psi|)^{1/2} \tag{3.9}$$

(see equation (2.1)) where ψ is the "co-latitude", so that δ increases with increasing distance from the poles. Owing to this ψ-dependence of δ, at a given distance r from the axis the boundary layer on the outer sphere is thinner than the layer on the inner sphere and therefore transports less fluid towards the equator. In contrast to the cylindrical case, continuity demands an axial flow u_z in the inviscid interior, where u_z is independent of the axial coordinate (in keeping with the third component of equation (1.15)) and aries with r according to the expression

$$u_z(r) = \frac{Q}{4\pi \hat{b}^2 \hat{a}^2}\left\{\frac{\hat{a}^2\left(1-\frac{r^2}{\hat{a}^2}\right)^{1/4}\left(1-\frac{r^2}{\hat{b}^2}\right)^{-3/4} - \hat{b}^2\left(1-\frac{r^2}{\hat{b}^2}\right)^{1/4}\left(1-\frac{r^2}{\hat{a}^2}\right)^{-3/4}}{\left(1-\frac{r^2}{\hat{a}^2}\right)^{1/4} + \left(1-\frac{r^2}{\hat{b}^2}\right)^{1/4}}\right\} \qquad (3.10)$$

hen $r < \hat{a}$, the azimuthal motion being related to u_z by the Ekman suction ormula given by equation (2.4). A striking feature of the flow is the absence n this geostrophic limit of any motion in the fluid occupying the region between he imaginary cylindrical surface $r = \hat{a}$ and the "low-latitude" part of the outer ounding sphere which extends from $r = \hat{a}$ to the "equator" at $r = \hat{b}$. The geostrophic zimuthal flow in the region $r < \hat{a}$ drops to zero at $r = \hat{a}$ but its rate of change ith respect to r does not, implying that a weak ageostrophic detached shear ayer will be present near $r = \hat{a}$. This layer and the boundary layers on the ylindrical surfaces of the source and sink as well as the boundary layer at the quator of the inner sphere, where Ekman theory breaks down (as evinced by the behaviour f the right-hand side of equation (3.9) when $\psi = \pi/2$) are complex in structure nd their theoretical investigation poses some very difficult mathematical problems. gain, however, the flow elsewhere can be determined by elementary theoretical onsiderations and unpublished experiments carried out in my laboratory demonstrate onclusively that this flow occurs in practice. (The study of source-sink flows is ot, of course, directly relevant to dynamical meteorology, but it is nevertheless nteresting to note that the tendency for the main constituent of the Martian tmosphere, carbon dioxide, to freeze out near the winter pole gives rise to a et poleward atmospheric flow and a concomitant increase in azimuthal wind speed).

.2 Non-axisymmetric source-sink flow. The simplest conceivable non-axisymmetric ystem is the two dimensional system discussed first in § 3.1 above with one ery simple modification, namely the insertion at azimuth $\phi = \phi_0$ (say) of a hin rigid impermeable radial barrier. The velocity field is then determined irtually everywhere by considerations of continuity (see equation (1.1)) when viscous ffects are confined to thin boundary layers on the radial barrier. Thus

$$\underset{\sim}{u} = (u_r, u_\phi, u_z) = (q/2\pi r, 0, 0) \qquad (3.11)$$

nd therefore independent of Ω (cf equation (3.1)) and $\mathrm{curl}\, \underset{\sim}{u} \equiv \underset{\sim}{\xi} = 0$
cf. equation (3.3)). The pressure satisfies

$$\frac{\partial P}{\partial \phi} = \frac{\Omega q}{\pi} \qquad (3.12)$$

mplying that a pressure difference $2\pi \bar{\rho} q$ develops across the radial barrier n $\phi = \phi_0$.

Owing to the absence of relative vorticity in the basic two-dimensional flow iven by equation (3.11) (in contrast to the case discussed in §3.1), the flow ill be largely unaffected by boundary layers on end-walls which are everywhere erpendicular to the rotation axis or are axisymmetric and such that $\nabla_1(z_u - z_\ell) = 0$ verywhere. The end-walls are then passive in the sense that they merely reduce he relative flow from its non-zero value in the interior region to zero on the all, where the no-slip boundary condition must be satisfied.

When the axial distance D (see equation (3.4)) is non-uniform and the system s non-axisymmetric it is necessary to consider topographic end effect. We have een in §1.5 that in quasi-geostrophic flow of a homogeneous incompressible fluid changes in relative vorticity are brought about mainly by axial stretching and when, in addition to Ekman suction, end-wall topography contributes to axial stretching, the axial component of the quasi-geostrophic relative vorticity, ζ, satisfies equation (2.5).

The relative importance of the topographic contribution to vorticity changes, as represented by the term $2\Omega D^{-1}(\underset{\sim}{u}_1 . \nabla_1)D$ in equation (2.5), is measured by the ratio h/h_* where h is the amplitude of variations in D and

$$h_* \sim \epsilon D + \delta . \tag{3.13}$$

Topographic end effects will not be important when $h < h_*$, but when $h > h_*$, — and this is always the case for strictly geostrophic motion since h_* then vanishes — such effects are so strong that within the main body of the fluid the flow is steered (to $O(\epsilon)$) along geostrophic contours, defined as curves on which

$$D \equiv z_u - z_e \quad = \quad \text{constant.} \tag{3.14}$$

Another solution of the "steering equation"

$$(\underset{\sim}{u}_1 . \nabla_1)D = 0 \tag{3.15}$$

satisfied by $\underset{\sim}{u}_1$ is $\underset{\sim}{u}_1 = 0$ and there <u>are</u> circumstances in which the effect of topography is to produce stagnation, as in the case of the equatorial region of the spherical system discussed in § 3.1 above. Quasi-geostrophic motion is clearly impossible in regions where, owing to the geometry of the end-walls, continuous geostrophic contours cannot be found, and within such regions the flow, if it does not vanish, either oscillates rapidly or is characterized by strong transverse shear.

The effect of axisymmetric sloping end-walls on the source-sink flow here under consideration (see equation (3.11)) is particularly striking — in contrast to the case discussed in § 3.1 above, for which the flow given by equation (3.5) is everywhere parallel to the geostrophic contours and is therefore unaffected by topographic stretching. Figure 8a of Hide (1977) illustrates the case when D increases with increasing distance from the axis (ie $dD/dr > 0$) and $D(b) - D(a) \gg h_*$ (see equations (3.13) and (3.15)), and figure 8b the case when $dD/dr < 0$ and $D(a) - D(b) \gg h_*$. In the main body of the fluid, there can be no flow across geostrophic contours, which are circles concentric with the axis of rotation, and , owing to this major constraint on the flow, motion is largely confined to highly ageostrophic boundary layers on the cylindrical surfaces on the source and sink and on one side or the other side of the radial barrier, where the transfer of fluid from the inner cylinder to the outer cylinder (when $q > 0$, the case illustrated) takes place in a "western boundary current" when $dD/dr > 0$ (see Figure 8a) or an "eastern boundary current" when $dD/dr < 0$ (see Fibure 8b). The motion simply reverses direction, with no significant change in the general flow pattern, when $q < 0$, corresponding to the case when fluid enters the system via the outer cylinder rather than the inner cylinder. Within the "western" or "eastern" boundary current, the "planetary vorticity" term $2\Omega D^{-1}(u_1.\nabla_1)D$ is balanced by the sum of the non-linear advective term $(\underset{\sim}{u}_1 . \nabla_1)\zeta$ and the viscous terms in the vorticity equation (see equations (1.6) and (2.5)).

It is possible to show, incidentally, that in its main dynamical effects the sloping end-walls with D increasing outwards is formally equivalent in the case of a homogeneous fluid to the latitudinal variation of the Coriolis parameter f (the vertical component of 2Ω, see equation (1.23)) when dealing with flow in a thin spherical shell (for references see Greenspan 1968). This is often called the "beta-effect" in dynamical meteorology and oceanography, owing to the use of the so-called "beta-plane" where local Cartesian coordinates are used (with the x-axis towards the east and the y-axis towards the north) and is taken as a linear function of y;

$$f = f_0 + \beta y \, . \tag{3.16}$$

The best known example of a western boundary current in nature is the Gulf Stream in the Atlantic Ocean (Stommel 1965, Veronis 1973), the ~~earliest~~ theoretical studies of which were greatly aided by investigations of cylindrical source-sink systems akin to those we have just discussed (see Hide 1977). Inserting a full meridional barrier connecting the source to the sink in the spherical system discussed in § 3.1 gives rise to a cross-equatorial western boundary current reminiscent of the East-African low-level cross-equatorial jet stream in the atmosphere and of the Somali current in the Indian Ocean.

3.3 Other systems. Time and space do not permit the detailed treatment of further examples of systems which illustrate directly the importance of the role played by boundary layers in various fundamental processes in rotating fluids. One of the best known of these processes is "spin-up" (see equation (2.6)), which has received a great deal of attention in the literature and has been the subject of several excellent reviews (see e.g. Benton and Clark 1974, Buzyna and Veronis 1971, Greenspan 1968). Central to the understanding of these processes in the case of a homogeneous fluid is the Proudman theorem expressing, effectively, the tendency for slow disturbances to propagate preferentially in directions parallel to the rotation axis.

The source-sink system discussed in § 3.1 is a convenient one for studying the disturbance produced by a localized bump on one of the end-walls or by a solid object suspended within the main body of the fluid. The wake due to the presence of such an obstacle to the flow takes the form of a "Taylor column" trailing at an angle $\sim \epsilon$ to the z axis when $\epsilon \ll 1$. In the "viscous" limit when $\epsilon \ll E^{1/4} (\ll 1)$ the column is parallel to the z axis, the flow within it is virtually stagnant, and the "walls" of the column are highly ageostrophic detached shear layers of thickness $\sim D^{1/2}\delta^{1/2}$. Otherwise, ie when $E^{1/4} \ll \epsilon \; (\ll 1)$ the Taylor column is of the so-called "inertial" type and much more complicated than that found in the viscous limit.

Recent studies of "spin-up" and Taylor columns include work on effects due to density stratification. Various lines of theoretical and experimental evidence indicate that stratification restricts the penetration distance parallel to the rotation axis to a value

$$\sim \Omega L_{\perp} / \pi N \tag{3.17}$$

where $L_{\perp}$ is a typical linear dimension transverse to the rotation axis and N is the Brunt-Väisälä "buoyancy" frequency. The ratio of this penetration distance to a typical axial dimension of the system is a fundamental parameter in the study of rotating non-homogeneous fluids and when this parameter is small (but $\epsilon \ll 1$ and $E \ll 1$) the side-wall boundary layers discussed in § 2.2 (see e.g. Benton and Clark 1974) and other features of the flow (see e.g. Lineykin 1974) — but not the Ekman layers — are controlled by the combined effects of rotation and stratification. Many of the papers listed in the bibliography (see Appendix A) deal with various aspects of boundary layers in rotating non-homogeneous fluids, including effects due to time variations in the basic flow, and the reader is referred to these papers for further details.

APPENDIX A

BIBLIOGRAPHY ON BOUNDARY LAYERS IN ROTATING FLUIDS AND ALLIED TOPICS

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